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ALGEBRAIC STABILITY AND DEGREE GROWTH OF MONOMIAL MAPS AND POLYNOMIAL MAPS

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Abstract. Given a rational monomial map, we consider the question of finding a toric variety on which it is algebraically stable. We give conditions for when such variety does or does not exist. We also obtain several precise estimates of the degree sequences of monomial maps on $\mathbb{P}^n$. Finally, we characterize polynomial maps which are algebraically stable on $(\mathbb{P}^1)^n$.

Contents

1. Introduction

We study the dynamical behavior of two family of maps, namely, monomial maps and polynomial maps. In particular, we focus on two aspects: algebraic stability and degree growth. For monomial maps, we use the theory of toric varieties as the main tool. For polynomial maps, we focus on their dynamical behavior on the space $(\mathbb{P}^1)^n = \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{n}$.

Given an $n \times n$ integer matrix $A = (a_{i,j})$, there is an associated monomial map $f_A : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$f_A(x_1, \cdots, x_n) = (\prod_j x_j^{a_{1,j}}, \cdots, \prod_j x_j^{a_{n,j}}).$$

Monomial maps fit nicely into the framework of toric varieties and equivariant maps (also called toric maps) on them. In this paper, we try to make extensive use of the toric method to study the dynamics of monomial maps.

The idea of applying the theory of toric varieties to monomial maps is in fact not new. For example, Favre [3] used the orbit-cone correspondence of the torus action to translate a criterion of algebraic stability to a condition about cones in a fan, and uses it to classify monomial maps in the case of toric surfaces. In order to generalize his result to higher dimension, one needs a good understanding on pulling back cohomology classes under rational maps. So we start from a formula of pulling back divisors in toric varieties (Theorem 4.1).

We then define the notion of algebraic stability and prove a criterion similar to the one in [3]. Results about stability are proven using the criterion. For example, we proved that every monomial polynomial map is algebraically stable on $(\mathbb{P}^1)^n$. Also, we generalize some results of [3] to higher dimension.

Theorem 5.7. *Suppose that $A \in \mathbf{M}_n(\mathbb{Z})$ is an integer matrix.*

1. *If there is a unique eigenvalue λ of A of maximal modulus, with algebraic multiplicity one; then $\lambda \in \mathbb{R}$, and there exists a simplicial toric birational model X (maybe singular) and a $k \in \mathbb{N}$ such that f_A^k is strongly algebraically stable on X.*
2. *If $\lambda, \bar{\lambda}$ are the only eigenvalues of A of maximal modulus, also with algebraic multiplicity one, and if $\lambda = |\lambda| \cdot e^{2\pi i \theta}$, with $\theta \notin \mathbb{Q}$; then there is no toric birational model which makes f_A strongly algebraically stable.*

For the definition of (strongly) algebraically stable, see section 5. We note that many of the results concerning stabilization of monomial maps in this paper have been obtained independently by Mattias Jonsson and Elizabeth Wulcan [9].

Next, we focus on two spaces: the projective space $\mathbb{P}^n$, and the product of the projective line $(\mathbb{P}^1)^n$. For the projective space $\mathbb{P}^n$, the monomial map f_A induces a rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$, also denoted by f_A. The pull back f^* of a rational map $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ on $H^{1,1}(\mathbb{P}^n; \mathbb{R})$ is given by the degree of

f. Thus we consider the degree sequence $\{\deg(f_A^k)\}_{k=1}^{\infty}$. Results about the degree sequence of monomial maps can be found in [1] and [8].

In particular, one can define the *asymptotic degree growth*

$$\delta_1(f_A) = \lim_{k\to\infty} (\deg(f_A^k))^{\frac{1}{k}}.$$

Hasselblatt and Propp ([8, Theorem 6.2]) proved that $\delta_1(f_A) = \rho(A)$, the spectral radius of the matrix A. We refine the above result and obtain the following description of the asymptotic behavior of the degree sequence for a general monomial map.

Theorem 6.2. *Given an $n \times n$ integer matrix A with nonzero determinant, assume that $\rho(A)$ is the spectral radius of A. Then there exist two positive constants $C_1 \geq C_0 > 0$ and a unique integer ℓ with $0 \leq \ell \leq n-1$, such that*

$$C_0 \cdot k^{\ell} \cdot \rho(A)^k \leq \deg(f_A^k) \leq C_1 \cdot k^{\ell} \cdot \rho(A)^k$$

for all $k \in \mathbb{N}$.

In fact, $(\ell + 1)$ is the size of the largest Jordan block of A among the ones corresponding to eigenvalues of maximal modulus.

Moreover, if the matrix A has some better property, then we can describe the degree sequence even more precisely. This is the content of Theorem 6.6, Theorem 6.7, and the following theorem.

Theorem 6.8. *Assume that the matrix A is diagonalizable, and assume for each eigenvalue λ of A of maximum modulus, $\lambda/\bar{\lambda}$ is a root of unity. Then there is a positive integer p, and p constants $C_0, C_1, \cdots, C_{p-1} \geq 1$, such that*

$$\deg(f_A^{pk+l}) = C_l \cdot |\lambda_1|^{pk+l} + O(|\lambda_2|^{pk+l}),$$

where $l = 0, 1, \cdots, p-1$.

Let us mention that the above theorems about the degree sequences of monomial maps can be generalized to the case of weighted projective spaces. On weighted projective spaces, we have the notion of *weighted degree* of a toric map, and their growth under iterations follows the pattern as the degree growth of monomial maps in projective spaces. This generalization is suggested to us by Mattias Jonsson. We introduce weighted projective space briefly, and explain the generalization in §6.3.

On $(\mathbb{P}^1)^n$, we obtain a concrete matrix representation for the pull back on the Picard groups for general rational maps. We apply the matrix representation to give another proof of the above theorem of Hasselblatt and Propp about the first dynamical degree of a monomial map (Theorem 7.1).

In the last subsection (§7.5) of this paper, we study the stability of polynomial maps on $(\mathbb{P}^1)^n$. As a result, we obtain the following characterization:

Theorem 7.5. *Let $f = (f_1, \cdots, f_n)$ be a polynomial map.*

1. *If each f_j is dominated by a monomial term, the f is algebraically stable on $(\mathbb{P}^1)^n$.*

(2) *Assume that, for some iterate* $f^N = (f_1^{(N)}, \cdots, f_n^{(N)})$ *of* f*, we have* $\deg_{z_i}(f_j^{(N)}) > 0$ *for all* $i, j = 1, \cdots, n$*. Then* f *being algebraically stable on* $(\mathbb{P}^1)^n$ *implies that each* f_j *must have a dominant term.*

Here we say that a polynomial $f_j(z_1, \cdots, z_n)$ is dominated by the monomial μ if the coefficient of μ in f_j is non-zero, and $\deg_{z_i}(f_j) = \deg_{z_i}(\mu)$ for all variables z_i, $i = 1, \cdots, n$.

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2. Toric varieties

In this section, we give a brief survey of basic definitions and properties of toric varieties. For more detail, we refer the readers to [2] or [6].

2.1. Cones and affine toric varieties.

Let $N \cong \mathbb{Z}^n$ be a lattice of rank n, and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ be the associated real vector space. A (convex) *polyhedral cone* in $N_{\mathbb{R}}$ is a subset of the form

$$\sigma = \{\sum_{i=1}^{k} r_i v_i \mid r_i \in \mathbb{R}_{\geq 0}, v_i \in N_{\mathbb{R}}\}$$

for some finite set of vectors $v_1, \cdots, v_k$. In the case $k = 0$, we make the convention that $\sigma = \{0\}$, the cone containing only the origin.

The *dimension* of a cone is the dimension of the $\mathbb{R}$-linear subspace spanned by the generating set. A cone is *strongly convex* if it does not contain any line through the origin. A cone is *rational* if we can choose the generators $v_1, \cdots, v_k$ from the lattice N. In what follows, by a cone we always mean "a strongly convex, rational polyhedral cone".

From the lattice N we can form the *dual lattice* $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, with dual pairing denoted by $\langle \ , \ \rangle$. It is a lattice in the dual vector space

$$M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{Hom}_{\mathbb{R}}(N_{\mathbb{R}}, \mathbb{R}) = N_{\mathbb{R}}^{\vee}.$$

The *dual cone* $\sigma^{\vee}$ of σ is defined by

$$\sigma^{\vee} = \{u \in M_{\mathbb{R}} \mid \langle u, v\rangle \geq 0 \text{ for all } v \in \sigma\}.$$

The intersection $S_\sigma = \sigma^{\vee} \cap M$ is a finitely generated monoid by Gordan's lemma. The affine variety $U_\sigma := \mathrm{Spec}(\mathbb{C}[S_\sigma])$ of the ring $\mathbb{C}[S_\sigma]$ is called the *affine toric variety* associated to the cone σ. More concretely, a closed point in U_σ corresponds to a semigroup morphism $(S_\sigma, +) \to (\mathbb{C}, \cdot)$ which sends $0 \in S_\sigma$ to $1 \in \mathbb{C}$.

Example 2.1. Let $N = \mathbb{Z}^2$, and let σ be the cone in $N_{\mathbb{R}} \cong \mathbb{R}^2$ generated by $e_1 = (1, 0)$ and $e_2 = (0, 1)$. It is easy to see that S_σ is the monoid generated

by the dual basis e_1^*, e_2^*, and $\mathbb{C}[S_\sigma] \cong \mathbb{C}[x, y]$. Thus the affine toric variety $U_\sigma \cong \operatorname{Spec} \mathbb{C}[x, y] = \mathbb{C}^2$.

Example 2.2. More generally, let $N = \mathbb{Z}^n$, and let σ be the cone in $N_\mathbb{R} \cong \mathbb{R}^n$ generated by the standard basis $e_1, \cdots, e_n$. Then the affine toric variety $U_\sigma \cong \mathbb{C}^n$.

Now we are going to introduce some definitions about cones.

- One dimensional cones are also called *rays*. On each ray, there is a unique nonzero integral point of the smallest norm ; it is called *the ray generator*.
- A cone is *simplicial* if it is generated by linearly independent vectors.
- A cone is *smooth* if it is generated by part of a basis for the lattice N.

We remark here that a cone σ is smooth if and only if the corresponding affine variety U_σ is smooth.

2.2. **Fans and general toric varieties.** A fan Σ in $N_\mathbb{R}$ is a set of cones in $N_\mathbb{R}$ satisfying the following two conditions:

(1) each face of a cone in Σ is a cone in Σ.
(2) the intersection of two cones in Σ is a face of each (hence also in Σ).

Unless otherwise stated, we will assume that fans are *finite*, i.e., they contain finitely many cones.

From a fan Σ, we can construct the *toric variety* $X(\Sigma)$ corresponding to Σ. First, we take the disjoint union $\coprod_{\sigma\in\Sigma} U_\sigma$, then we glue them as follows. For cones $\sigma, \tau \in \Sigma$, the intersection $\sigma \cap \tau$ is a face of each. Thus we have open immersions

$$U_\sigma \hookleftarrow U_{\sigma\cap\tau} \hookrightarrow U_\tau.$$

For each pair of cones $\sigma, \tau \in \Sigma$, we glue U_σ and U_τ along the open subvariety $U_{\sigma\cap\tau}$. The gluing data will be compatible, and the resulting variety is the toric variety $X(\Sigma)$.

We use the notion $\Sigma(k)$ to denote the set of all k-dimensional cones of Σ. The *support* of a fan Σ, denoted $|\Sigma|$, is the union of its cones, i.e., $|\Sigma| := \cup_{\sigma\in\Sigma}\sigma$. A fan Σ is *complete* if $|\Sigma| = N_\mathbb{R}$. A fan is complete if and only if the corresponding toric variety is a complete variety, i.e., the underlying topological space is compact in the classical topology. For a fan Σ, the toric variety $X(\Sigma)$ is smooth if and only if every cone in Σ is smooth. To check smoothness, it is enough to check for all *maximal cones* in Σ, i.e., cones that are not proper faces of any other cone.

Example 2.3. Let $N = \mathbb{Z}^2$, and let $e_0 = (-1, -1) = -(e_1 + e_2)$. Let τ_i be the ray generated by e_i for $i = 0, 1, 2$, and σ_{ij} be the cone generated by e_i and e_j. The set

$$\Sigma = \{\ \{0\}, \tau_0, \tau_1, \tau_2, \sigma_{12}, \sigma_{02}, \sigma_{01}\ \}$$

is a fan, and the corresponding toric variety is the projective plane $\mathbb{P}^2$. In fact, if we set the homogeneous coordinate of $\mathbb{P}^2$ as $[w; x; y]$, then we

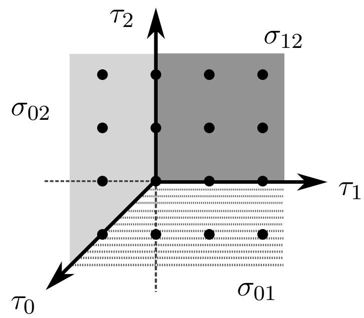

FIGURE 1. The fan structure of $\mathbb{P}^2$.

can identify $U_{\sigma_{12}}$ as $\operatorname{Spec}\mathbb{C}[\frac{x}{w},\frac{y}{w}] \cong \{[w;x;y] \in \mathbb{P}^2 | w \neq 0\}$. Similarly, $U_{\sigma_{02}} \cong \operatorname{Spec}\mathbb{C}[\frac{y}{x},\frac{w}{x}] \cong \{[w;x;y] \in \mathbb{P}^2 | x \neq 0\}$, and $U_{\sigma_{01}} \cong \operatorname{Spec}\mathbb{C}[\frac{x}{y},\frac{w}{y}] \cong \{[w;x;y] \in \mathbb{P}^2 | y \neq 0\}$.

Example 2.4. More generally, let $N = \mathbb{Z}^n$, $e_1, \cdots, e_n$ be the standard basis of N, and $e_0 = -(e_1 + \cdots + e_n)$. For any proper subset I of the set $\mathbf{n} = \{0, 1, \cdots, n\}$, i.e., $I \subsetneq \mathbf{n}$, let σ_I be the cone generated by $\{e_i | i \in I\}$. Then the set $\Sigma = \{\sigma_I | I \subsetneq \mathbf{n}\}$ forms a fan. The toric variety associated to the fan is the projective space $X(\Sigma) \cong \mathbb{P}^n$.

Example 2.5. In this example, we will construct a fan Σ that corresponds to the product of the projective line $(\mathbb{P}^1)^n = \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{n}$. There are $2n$ rays in Σ. They are generated by the standard basis vectors $e_1, \cdots, e_n$ and their negatives $-e_1, \cdots, -e_n$. The maximal cones, i.e., n-dimensional cones, are generated by vectors of the form $\{(s_1 e_1), \cdots, (s_n e_n)\}$, where $s_i \in \{+1, -1\}$ are the signs. All other cones in Σ are faces of some maximal cone. For example, the fan for $\mathbb{P}^1 \times \mathbb{P}^1$ has four maximal cones, generated by $\{e_1, e_2\}, \{-e_1, e_2\}, \{e_1, -e_2\}$, and $\{-e_1, -e_2\}$, respectively.

2.3. **The orbits of the torus action.** Every toric variety $X(\Sigma)$ is equipped with a torus action, thus $X(\Sigma)$ can be written as the disjoint union of the orbits. The orbits are in 1-1 correspondence with cones in the fan Σ in a nice way, as follows. First, for each cone $\tau \in \Sigma$, there is a distinguished point $x_\tau \in U_\tau$. It is the closed point corresponding to the following semigroup morphism $S_\sigma \to \mathbb{C}$.

$$u \longmapsto \begin{cases} 1 & \text{if } u \in \sigma^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

Then we define O_τ to be the orbit of x_τ under the torus action. The only closed orbits are fixed points, which correspond to the n-dimensional cones. We can take the closure of an orbit in the toric variety, it is called the *orbit closure* of τ, denoted by $V(\tau)$. An orbit closure itself is a toric variety, it is a closed, toric subvariety of the original toric variety.

3. Toric maps

Suppose $A : N \to N'$ is a homomorphism of lattices, Σ is a fan in N, and Σ' is a fan in N'.

Definition. Given a cone $\sigma \in \Sigma$, we say that σ *maps regularly* to Σ' by A if there is a cone $\sigma' \in \Sigma'$ such that $A(\sigma) \subseteq \sigma'$. In this case, we call the smallest such cone in Σ' the *cone closure* of the image of σ, and denote it by $\overline{A(\sigma)}$.

If $A : N \to N'$ is a homomorphism such that every cone of Σ maps regularly to Σ', then A induces a morphism of varieties $f_A : X(\Sigma) \to X(\Sigma')$. Furthermore, f_A will be *equivariant* under the torus action. Conversely, every equivariant morphism $X(\Sigma) \to X(\Sigma')$ is induced by a homomorphism of lattices satisfying the above property. Equivariant morphisms will map orbits to orbits. If $\sigma \in \Sigma$, then f_A maps $O_\sigma \subset X(\Sigma)$ to $O_{\overline{A(\sigma)}} \subset X(\Sigma')$.

More generally, any homomorphism of lattices $A : N \to N'$ induces an *equivariant rational map* $f_A : X(\Sigma) \dashrightarrow X(\Sigma')$. On a complete toric variety, f_A is *dominant* if and only if $A_\mathbb{R} = (A \otimes \mathbb{R}) : N_\mathbb{R} \to N'_\mathbb{R}$ is surjective. In this paper, we will study the dynamics of dominant, toric rational self maps on complete toric varieties.

Example 3.1. Let the matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z})$, and identify $N \cong \mathbb{Z}^2$ as column vectors. Then $A : N \to N$ is given by multiplying column vectors by A on the left. Let us see what f_A does on the affine toric variety $\mathbb{C}^2$ we constructed in Example 2.1. Notice that A induces the linear map ${}^tA$ on M, which sends $e_1^* \mapsto a \cdot e_1^* + b \cdot e_2^*$ and $e_2^* \mapsto c \cdot e_1^* + d \cdot e_2^*$. The isomorphism $\mathbb{C}[S_\sigma] \cong \mathbb{C}[x, y]$ is given by $e_1^* \mapsto x$ and $e_2^* \mapsto y$, thus f_A is the map $x \mapsto x^a y^b$ and $y \mapsto x^c y^d$.

Example 3.2. Let $A = (a_{i,j})$ be an $n \times n$ integer matrix, then using a similar argument as the above example, we can see that the map $f_A : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ is given by

$$f_A(x_1, \cdots, x_n) = (\textstyle\prod_j x_j^{a_{1,j}}, \cdots, \prod_j x_j^{a_{n,j}}).$$

So we know on $\mathbb{C}^n$, toric rational self maps are exactly monomial maps.

3.1. **Toric endomorphisms.** Before we start to study the dynamics of toric rational self maps, one might ask: what do we know about toric morphisms from a toric variety to itself? The following property provides an answer. It turns out that those morphisms have quite simple structure.

Let Σ be a complete fan in $N_\mathbb{R}$. Let $A : N \to N$ be a homomorphism of lattice that maps each cone of Σ regularly to Σ, then $f_A : X(\Sigma) \to X(\Sigma)$ is a toric morphism. Also assume that f_A is dominant, i.e. $A_\mathbb{R} : N_\mathbb{R} \to N_\mathbb{R}$ is surjective. Moreover, suppose that $v_1, \cdots, v_d$ are the ray generators of Σ.

Proposition 3.1. *There is a positive integer k such that $A^k v_i = a_i v_i$ for some positive integers a_i, $i = 1, \cdots, d$. In particular, the iteration $f_A^k : X(\Sigma) \to X(\Sigma)$ maps every orbit onto itself.*

Proof. We know that A maps each cone σ into some other cone σ'. Since $A_{\mathbb{R}}$ is a surjective endomorphism of $N_{\mathbb{R}}$, it is indeed an $\mathbb{R}$-linear automorphism. Thus $A_{\mathbb{R}}$ will preserve the dimension of cones in Σ. As a consequence, what $A_{\mathbb{R}}$ does on the cones of Σ is just permuting them (preserving the dimension). Therefore, for some integer k, $A_{\mathbb{R}}^k$ would fix every cone in Σ. In particular, for one dimensional cones, we know that $A_{\mathbb{R}}^k(\mathbb{R}_{\geq 0} \cdot v_i) = \mathbb{R}_{\geq 0} \cdot v_i$. Furthermore, since $A : N \to N$ and $v_i \in N$, we deduce that $A(v_i) = a_i v_i$ for some positive integer n_i. Finally, the last sentence is an immediate consequence of the first sentence and the description of images of orbits under a morphism. □

Example 3.3. Given non-zero integers a_i, the endomorphism of $(\mathbb{P}^1)^n = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ coming from the monomial map of $\mathbb{C}^n$ of the form $f(x_1, x_2 \cdots, x_n) = (x_1^{a_1}, x_2^{a_2}, \cdots, x_n^{a_n})$ is toric. Conversely, every toric endomorphism of $(\mathbb{P}^1)^n$ is of the form

$$f(x_1, x_2 \cdots, x_n) = (x_{s(1)}^{a_1}, x_{s(2)}^{a_2}, \cdots, x_{s(n)}^{a_n})$$

for some permutation $s \in S_n$.

The above example shows that, in general, the above proposition cannot be improved. However, if the space $X(\Sigma)$ is nice, there are possibly stronger condition on the map A, as we can see in the following example.

Example 3.4. For the space $\mathbb{P}^n$, we know the one dimensional cones are generated by $e_0, e_1 \cdots, e_n$. Every n of them would form a basis for N, and $e_0 + e_1 + \cdots + e_n = 0$. Applying A^k, we get $a_0 e_0 + a_1 e_1 + \cdots + a_n e_n = 0$. This will force $a_0 = a_1 = \cdots = a_n$, and hence we know $f_A^k([x_0 : \cdots : x_n]) = [x_0^a : \cdots : x_n^a]$ for some positive integer a.

4. Divisors on toric varieties

Divisors are the main tool in the study of codimension-one geometry of varieties. Since we work on toric varieties, the divisors that are invariant under the torus action are especially important. We will recall basic definitions and properties of divisors in a toric variety, then prove a formula about pulling back divisors.

4.1. **Weil Divisors and divisor class groups.** In a toric variety, the torus invariant prime Weil divisors are exactly the codimension one orbit closures, i.e., $V(\tau)$ for $\tau \in \Sigma(1)$. Let $\Sigma(1) = \{\tau_1, \cdots, \tau_d\}$ be the (finite) set of rays in Σ, a T-invariant Weil divisor, T-Weil divisor for short, is then of the form $\sum_{i=1}^d a_i V(\tau_i)$ where $a_i \in \mathbb{Z}$. The group of T-Weil divisors, denoted $\mathrm{WDiv}_T(X(\Sigma))$, then equals to $\oplus_{i=1}^d \mathbb{Z} \cdot V(\tau_i)$, the free abelian group generated by the T-invariant prime divisors. The principal divisors in $\mathrm{WDiv}_T(X(\Sigma))$ are in 1-1 correspondence to elements of M in the following way. For each element $u \in M$, there is a corresponding character $(\mathbb{C}^*)^n \to \mathbb{C}^*$ which extend to a global meromorphic function χ^u on $X(\Sigma)$. This gives the principal

divisor

$$\operatorname{div}(\chi^u) = \sum_{i=1}^{d} \langle u, v_i \rangle V(\tau_i),$$

where v_i is the ray generator of τ_i. Conversely, every principal divisor in $\mathrm{WDiv}_T(X(\Sigma))$ is of the form $\operatorname{div}(\chi^u)$ for some $u \in M$. Hence we can identify M as a subgroup of $\mathrm{WDiv}_T(X(\Sigma))$, and the quotient

$$A_{n-1}(X(\Sigma)) := \mathrm{WDiv}_T(X(\Sigma))/M$$

is the *divisor class group* of $X(\Sigma)$.

4.2. **Cartier divisors and Picard groups.** In a complete toric variety $X(\Sigma)$ associated to a complete fan Σ, the torus invariant Cartier divisors, or for simplicity, T-Cartier divisors, is given by the following data. For each cone $\sigma \in \Sigma(n)$ of maximal dimension, we specify an element $u(\sigma) \in M$. The datum $\{u(\sigma) | \sigma \in \Sigma(n)\}$ are required to satisfy the compatibility condition that $[u(\sigma)] = [u(\sigma')]$ in $M/M(\sigma \cap \sigma')$, where $M(\sigma \cap \sigma') = (\sigma \cap \sigma')^{\perp} \cap M$. We write $D = \{u(\sigma)\}$ and call it the Cartier divisor defined by the data $\{u(\sigma)\}$. We denote the group of all T-Cartier divisor by $\mathrm{CDiv}_T(X(\Sigma))$.

Each $u(\sigma)$ defines a T-Weil divisor $\operatorname{div}(\chi^{-u(\sigma)})$ on U_σ (the negative sign here is to be consistent with the literature). The compatibility condition means these divisors agree on overlaps, thus every T-Cartier divisor $D = \{u(\sigma)\}$ gives rise to a unique T-Weil divisor

$$[D] = \sum_{\tau_i \in \Sigma(1)} -\langle u(\sigma), v_i \rangle \cdot V(\tau_i),$$

here σ in the summand is any maximal cone such that $\tau_i \in \sigma$. The sum is independent of the choice of σ by the compatibility condition.

The data $\{u(\sigma) | \sigma \in \Sigma(n)\}$ also defines a continuous piecewise linear function ψ_D on $N_{\mathbb{R}}$. The restriction of ψ_D to the maximal cone σ is given by $u(\sigma)$, i.e., $\psi_D(v) = \langle u(\sigma), v \rangle$ for $v \in \sigma$. The continuity comes from the compatibility. Conversely, a continuous piecewise linear function ψ on $N_{\mathbb{R}}$, which is also integral on each cone (i.e., given by an element of the lattice M on each cone), determines a unique T-Cartier divisor D, with $[D] = \sum -\psi(v_i) \cdot V(\tau_i)$. The function ψ is called the *support function* of the Cartier divisor D. On a complete toric variety, a T-Cartier divisor is *ample* if and only if its support function is strictly convex([6, p.70]).

There is a natural way to identify M as a subgroup of $\mathrm{CDiv}_T(X(\Sigma))$. Each $u \in M$ is identified with the Cartier divisor such that $u(\sigma) = u$ for all $\sigma \in \Sigma(n)$. The Weil divisor of this Cartier divisor is exactly the principal divisor defined by χ^u. The quotient $\mathrm{CDiv}_T(X(\Sigma))/M$ is the *Picard group* of $X(\Sigma)$, and is denoted by $\mathrm{Pic}(X(\Sigma))$.

We conclude this section by mentioning relations between Picard groups and cohomology groups. For a complete toric variety X, we have $\mathrm{Pic}(X) = H^2(X; \mathbb{Z})$. If X is also simplicial, then

$$H^{1,1}(X) := H^1(X, \Omega_X) = H^2(X; \mathbb{C}) = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}.$$

4.3. **Pulling back divisors.** Assume that Σ, Σ' are complete fans in $N_{\mathbb{R}}, N'_{\mathbb{R}}$, respectively, which associates to the complete toric varieties $X(\Sigma)$ and $X(\Sigma')$. Let A be a homomorphism $N \to N'$ such that $A \otimes_{\mathbb{Z}} \mathbb{R}$ is surjective. It induces a dominant rational map $f_A : X(\Sigma) \dashrightarrow X(\Sigma')$. We would like to study the pull back of a Cartier divisor on $X(\Sigma')$, which gives, in general, a Weil divisor of $X(\Sigma)$.

A Cartier divisor on $X(\Sigma')$ corresponds to a unique integral piecewise linear function on Σ'. Let D be a Cartier divisor, with support function ψ_D.

Theorem 4.1. *Let Σ, Σ' be complete fans, and $f_A : X(\Sigma) \dashrightarrow X(\Sigma')$ be a dominant toric rational map induced by $A : N \to N'$. The pull back of D via f_A is*

$$f_A^* D = \sum_{\tau_i \in \Sigma(1)} -\psi_D(Av_i) \cdot V(\tau_i).$$

Here the τ_i run through all one-dimensional cones of Σ, and v_i is the ray generator of the ray τ_i.

Proof. We can refine the fan Σ to get a fan $\widetilde{\Sigma}$ such that A induces a toric morphism from $X(\widetilde{\Sigma})$ to $X(\Sigma)$. In order to distinguish from f_A, we call this morphism $\widetilde{f}_A$. The morphism $\pi : X(\widetilde{\Sigma}) \to X(\Sigma)$ is induced by the identity map on N. It is proper and birational. So we have the following diagram.

$$\begin{tikzcd} & X(\widetilde{\Sigma}) \arrow[dl, "\pi"'] \arrow[dr, "\widetilde{f}_A"] & \\ X(\Sigma) \arrow[rr, dashed, "\widetilde{f}_A \circ \pi^{-1}"'] & & X(\Sigma') \end{tikzcd}$$

We are going to pull back the divisor D by first pull it back by $\widetilde{f}_A$, then push it forward by π, i.e., $f_A^* D = \pi_*(\widetilde{f}_A^* D)$. Notice that once we show f_A^* is given by the above expression, then since the expression is independent of the refined fan $\widetilde{\Sigma}$, so is f_A^*.

A Cartier divisor $D = \{u(\sigma); \sigma \in \Sigma(n)\}$ is locally cut out by the equation $\chi^{u(\sigma)} = 0$ on U_σ. The map $A : N \to N'$ induces the map $M' \to M$ defined by $u \mapsto u \circ A$. Therefore, the pull back of D under the morphism $\widetilde{f}_A$ is given by $\widetilde{f}_A^* D = \{u(\sigma) \circ A\}$. We can also describe the pull back $\widetilde{f}_A^* D$ using its support function. If ψ_D is the support function of D, then $\widetilde{f}_A^* D$ will have $\psi_D \circ A$ as its support function. Thus, as a Weil divisor, we have

$$\widetilde{f}_A^* D = \sum_{\tau_i \in \widetilde{\Sigma}(1)} -\psi_D(Av_i) \cdot V(\tau_i).$$

The fan $\widetilde{\Sigma}$ is a subdivision of Σ, and π is just the toric morphism induced by identity. The push forward map π_* is given, on the prime divisors $V(\tau)$

for $\tau \in \widetilde{\Sigma}(1)$, by

$$\pi_* V(\tau) = \begin{cases} V(\tau) & \text{if } \tau \in \Sigma(1), \\ 0 & \text{if } \tau \notin \Sigma(1). \end{cases}$$

Therefore, combining the two steps, we obtain

$$f_A^* D = \pi_*(\widetilde{f}_A^* D) = \sum_{\tau_i \in \Sigma(1)} -\psi_D(Av_i) \cdot V(\tau_i).$$

□

Notice that, if D is a principal divisor, i.e., ψ_D is a linear function, then the pull back $f_A^* D$ will again be principal, given by the linear function $\psi_D \circ A$. Thus it induces a map, also denoted by f_A^*, from the Picard group to the divisor class group.

$$f_A^* : \mathrm{Pic}(X(\Sigma')) \to A_{n-1}(X(\Sigma)).$$

If the fan is smooth, then so is the toric variety, and the notions of Weil divisors and Cartier divisors coincide. So the pull back map of a Cartier divisor is still Cartier, and it induces a map on Picard groups.

$$f_A^* : \mathrm{Pic}(X(\Sigma')) \to \mathrm{Pic}(X(\Sigma)).$$

In a simplicial toric variety, every Weil divisor D is $\mathbb{Q}$-Cartier, i.e., some positive integral multiple of D is Cartier. Thus if we denote $G_{\mathbb{Q}} = G \otimes_{\mathbb{Z}} \mathbb{Q}$ for an abelian group G, and assume that Σ, Σ' are simplicial, then we have both maps

$$f_A^* : \mathrm{CDiv}_T(X(\Sigma'))_{\mathbb{Q}} \to \mathrm{CDiv}_T(X(\Sigma))_{\mathbb{Q}},$$
$$f_A^* : \mathrm{Pic}(X(\Sigma'))_{\mathbb{Q}} \to \mathrm{Pic}(X(\Sigma))_{\mathbb{Q}}.$$

We use the same symbol f_A^* here to avoid inventing too many notations, but it has the drawback of making confusions. Thus we will state clearly whether we talk about divisors or divisor classes every time we use the symbol f_A^*.

What we do for pulling back $\mathbb{Q}$-Cartier divisors in the simplicial case is as follows. First, notice that an element $D \in \mathrm{CDiv}_T(X(\Sigma'))_{\mathbb{Q}}$ can be identified with a *rational* support function ψ_D, i.e., it takes rational values on the ray generators. The composition $(\psi_D \circ A)$ is piecewise linear on the fan $\widetilde{\Sigma}$, but not on Σ. We use the values of the function on rays to make an interpolation and obtain a piecewise linear function on Σ. This step is possible because Σ is simplicial. If we denote the modifying (interpolation) function by $\mu = \mu_{\widetilde{\Sigma},\Sigma}$, we can describe it more concretely. Let φ be a rational continuous piecewise linear function on $\widetilde{\Sigma}$, for a maximal cone $\sigma \in \Sigma(n)$, assume that $\tau_1 \cdots \tau_n$ are one-dimensional faces of σ, with ray generators $v_1, \cdots, v_n$, then $\mu(\varphi)|_\sigma = \sum_{i=1}^n \varphi(v_i) \cdot v_i^*$. Here v_i^* is the dual basis of v_i with respect to the basis $\{v_1, \cdots, v_n\}$.

To sum up, we have the following:

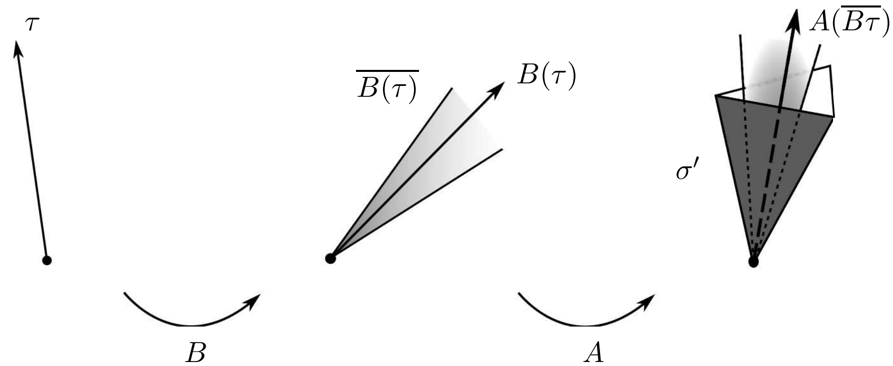

FIGURE 2. The geometric condition for $(f_A \circ f_B)^* = f_B^* \circ f_A^*$.

Corollary. *For complete, simplicial toric varieties $X(\Sigma), X(\Sigma')$, and a dominant toric rational map $f_A : X(\Sigma) \dashrightarrow X(\Sigma')$, we can write the procedure of pulling back divisors as $f_A^* D = \mu_{\widetilde{\Sigma},\Sigma}(\psi_D \circ A)$.* □

5. Algebraic stability

For the rest of this paper, all toric varieties are assumed to be complete and simplicial.

5.1. Definition and a geometric criterion.

Definition. A toric rational map $f_A : X(\Sigma) \dashrightarrow X(\Sigma)$ is *strongly algebraically stable* if $(f_A^k)^* = (f_A^*)^k$ as maps of $\mathrm{CDiv}_T(X(\Sigma))_{\mathbb{Q}}$ for all $k \in \mathbb{N}$. It is *algebraically stable* if $(f_A^k)^* = (f_A^*)^k$ as maps of $\mathrm{Pic}(X(\Sigma))_{\mathbb{Q}}$, for all k.

Notice that $(f_A^k)^* = (f_A^*)^k$ on $\mathrm{CDiv}_T(X(\Sigma))_{\mathbb{Q}}$ implies $(f_A^k)^* = (f_A^*)^k$ on $\mathrm{Pic}(X(\Sigma))_{\mathbb{Q}}$, so the condition for strongly algebraic stability is indeed stronger than that for algebraic stability. It is not clear to us whether the two conditions are equivalent or not in general. However, if we assume that the toric variety $X = X(\Sigma)$ is projective, then the two conditions are equivalent. We will prove that later in this section.

Our next goal is to prove a geometric characterization of strongly algebraically stable maps. We need to prove a lemma first. Given two homomorphisms of lattices $A : N \to N'$ and $B : N'' \to N$, they induce two toric rational maps $f_A : X(\Sigma) \to f(\Sigma')$ and $f_B : X(\Sigma'') \to f(\Sigma)$.

Lemma 5.1. *$(f_A \circ f_B)^* = f_B^* \circ f_A^*$ as maps*

$$\mathrm{CDiv}_T(X(\Sigma'))_{\mathbb{Q}} \to \mathrm{CDiv}_T(X(\Sigma''))_{\mathbb{Q}}$$

if and only if for each ray in Σ'', the cone closure of its image maps regularly to Σ'. That is, for each $\tau \in \Sigma''(1)$, there exists a $\sigma' \in \Sigma'$ such that $A(\overline{B(\tau)}) \subset \sigma'$.

Proof. First, suppose that the geometric condition is satisfied, we want to show $(f_A \circ f_B)^* = f_B^* \circ f_A^*$. Remember that $(f_A \circ f_B)^* D = \mu(\psi_D \circ (A \circ B))$ and $(f_B^* \circ f_A^*)D = f_B^*(f_A^* D) = \mu(\mu(\psi_D \circ A) \circ B)$, where μ is the modifying function. So it is enough to show that, for all $\tau_i \in \Sigma(1)$ and v_i the ray generator of τ_i,

$$(\psi_D \circ (A \circ B))(v_i) = (\mu(\psi_D \circ A) \circ B)(v_i),$$

that is, $\psi_D(A(Bv_i)) = \mu(\psi_D \circ A)(Bv_i)$.

Since $A(\overline{B(\tau_i)}) \subset \sigma'$ for some $\sigma' \in \Sigma'$ and ψ_D is linear on σ', hence $(\psi_D \circ A)$ is linear on $\overline{B(\tau_i)}$. The interpolation μ therefore does not do anything on $\overline{B(\tau_i)}$, and we have $\mu(\psi_D \circ A)(Bv_i) = (\psi_D \circ A)(Bv_i)$.

Conversely, if for some ray $\tau \in \Sigma(1)$, $\overline{B(\tau)}$ does not map regularly by A. This means that $A(\overline{B(\tau)})$ is not contained in any cone of Σ'. We will construct a divisor $D \in \mathrm{CDiv}_T(X(\Sigma'))$ such that $(f_A \circ f_B)^* D \neq (f_B^* \circ f_A^*)D$. Let $\gamma_1, \cdots, \gamma_m$ be the one-dimensional faces of Σ', and for $i = 1, \cdots, m$, let

$$a_i = \begin{cases} 0 & \text{if } \gamma_i \text{ is a face of } \overline{(A \circ B)(\tau)}, \\ 1 & \text{otherwise.} \end{cases}$$

Define $D = \sum_{i=1}^m a_i \cdot V(\tau_i')$, and let ψ_D be the support function of D. First, observe that $\psi_D(v) = 0$ if and only if $w \in \overline{(A \circ B)(\tau)}$. Thus for the divisor $(f_A \circ f_B)^* D$, the coefficient of $V(\tau)$ is 0.

On the other hand, since $A(\overline{B(\tau)})$ is not contained in $\overline{(A \circ B)(\tau)}$, there is some one dimensional face τ_0 of $\overline{B(\tau)}$ such that $A(\tau_0) \notin \overline{(A \circ B)(\tau)}$. Let v_0 be the ray generator of τ_0, then $\psi_D(Av_0) > 0$. Thus $\mu(\psi_D \circ A)$ is strictly positive in the relative interior of $\overline{B(\tau)}$, which contains Bv. This implies that the coefficient of $V(\tau)$ for the divisor $(f_B^* \circ f_A^*)D$ is strictly positive. Therefore we have $(f_A \circ f_B)^* D \neq (f_B^* \circ f_A^*)D$. □

Theorem 5.2. *A toric rational map $f_A : X(\Sigma) \dashrightarrow X(\Sigma)$ is strongly algebraically stable if and only if for all ray $\tau \in \Sigma(1)$ and for all $n \in \mathbb{N}$, $\overline{A^n(\tau)}$ maps regularly to Σ by A.*

Proof. First assume that $f = f_A$ is strongly algebraically stable. Thus for all $n \in \mathbb{N}$, we have $(f^n)^* = (f^*)^n$ and $(f^{n+1})^* = (f^*)^{n+1}$. This gives us

$$(f \circ f^n)^* = (f^{n+1})^* = (f^*)^{n+1} = (f^*)^n \circ f^* = (f^n)^* \circ f^*.$$

By the above lemma, the equality $(f \circ f^n)^* = (f^n)^* \circ f^*$ implies that $\overline{A^n(\tau)}$ is mapped regularly to Σ by f.

Conversely, assume that $\overline{A^n(\tau)}$ is mapped regularly to Σ by f for all $n \in \mathbb{N}$. This tells us that $(f \circ f^n)^* = (f^n)^* \circ f^*$ for all $n \in N$. Thus we have $(f^n)^* = (f^*)^n$ for all $n \in \mathbb{N}$ by an induction argument. □

In fact, let $\sigma_n = \overline{A^n(\tau)}$, the next lemma implies that, not only σ_n maps to Σ regularly, also the cone closure $\overline{A(\sigma_n)}$ is equal to $\sigma_{n+1} = \overline{A^n(\tau)}$.

Lemma 5.3. *Assume further that $A_{\mathbb{R}}$ is surjective, and $\overline{B(\tau)}$ maps regularly to Σ', then for all $\tau \in \Sigma''$, we have $\overline{A(\overline{B(\tau)})} = \overline{(A \circ B)(\tau)}$.*

That is, if σ is the smallest cone in Σ that contains $B(\tau)$, and σ' is the smallest cone in Σ' that contains $A(\sigma)$, then σ' will be the smallest cone in Σ' that contains $A(B(\tau))$.

Proof. Obviously, $\overline{(A \circ B)(\tau)}$ is a face of $\overline{A(\overline{B(\tau)})}$. Thus there is a supporting hyperplane H' of $\overline{A(\overline{B(\tau)})}$ in $N'_{\mathbb{R}}$ such that

$$\overline{A(\overline{B(\tau)})} \cap H' = \overline{(A \circ B)(\tau)}.$$

The preimage $H = A^{-1}(H')$ will then be a supporting hyperplane of $\overline{B(\tau)}$ in $N_{\mathbb{R}}$, so $\overline{B(\tau)} \cap H$ is a face of $\overline{B(\tau)}$ that contains $B(\tau)$. By the minimality of $\overline{B(\tau)}$, we must have $\overline{B(\tau)} \cap H = \overline{B(\tau)}$, i.e., $\overline{B(\tau)} \subset H$. Thus, $A(\overline{B(\tau)}) \subset H'$, and by the minimality of $\overline{A(\overline{B(\tau)})}$, we know $\overline{A(\overline{B(\tau)})} \subset H'$. Therefore,

$$\overline{A(\overline{B(\tau)})} = \overline{A(\overline{B(\tau)})} \cap H' = \overline{(A \circ B)(\tau)}.$$

□

With Theorem 5.2 and Lemma 5.3, we can describe the behavior, under iterations, of an strongly algebraically stable toric rational map f_A very concretely, as follows. For each ray $\tau \in \Sigma(1)$, let $\sigma_1 = \overline{A(\tau)}$ be the smallest cone containing $A(\tau)$, then σ_1 will map regularly to some cone in N, Assume $\sigma_2 = \overline{A(\sigma_1)} = \overline{A^2(\tau)}$ is the smallest such cone. Here the second equality is due to the lemma. Then σ_2 will map regularly again to some smallest $\sigma_3 = \overline{A(\sigma_2)} = \overline{A^2(\sigma_1)} = \overline{A^3(\tau)}$, and so on.

5.2. **Algebraic stable vs. strongly algebraic stable.** Now we can prove the equivalence of algebraic stable and strongly algebraic stable in the projective case. The equivalence of the two conditions, and a proof in the general case is mentioned to us by C. Favre. We adapted his proof to a proof for toric varieties.

Given two integer matrices $A, B \in \mathbf{M}_n(\mathbb{Z})$ with nonzero determinants, which induce two dominant toric rational maps $f_A, f_B : X \dashrightarrow X$.

Lemma 5.4. *Let D be an ample, T-invariant divisor on X, then the difference $(f_B^* \circ f_A^*)D - (f_A \circ f_B)^* D$ is an effective $\mathbb{Q}$-Cartier divisor.*

Proof. Write

$$(f_B^* \circ f_A^*)D = \sum_{\tau \in \Sigma(1)} a_\tau V(\tau), \quad (f_A \circ f_B)^* D = \sum_{\tau \in \Sigma(1)} b_\tau V(\tau).$$

We will show that $a_\tau \geq b_\tau$ for every $\tau \in \Sigma(1)$, which is equivalent to the lemma.

Let $\psi = \psi_D$ be the support function of D. For some $\tau \in \Sigma(1)$, let $v \in \tau$ be the ray generator. Let $\sigma = \overline{B\tau}$ be the smallest cone which contains $B\tau$,

and assume that $u_1, \cdots, u_d$ are the generators of the cone σ. Then there are positive numbers $r_1, \cdots, r_d$ such that $B(v) = r_1 u_1 + \cdots + r_d u_d$.

By the formula for pulling back divisors, to compute a_τ, we need to apply the *interpolation* process, and obtain

$$a_\tau = -[r_1 \psi(Au_1) + \cdots + r_d \psi(Au_d)].$$

We can also see that

$$b_\tau = -\psi((A \circ B)(v)) = -\psi(r_1 Au_1 + \cdots + r_d Au_d).$$

Now the fact $a_\tau \geq b_\tau$ comes from the fact that ψ is (strictly) convex since D is ample. $\square$

Proposition 5.5. *For a projective, complete, simple toric variety $X = X(\Sigma)$, a toric rational map f_A is strongly algebraically stable if and only if it is algebraically stable.*

Proof. Since strongly AS implies AS, it suffices to show the other direction. Assume that f_A is not strongly AS, then there is a ray τ and a positive integer k such that $A^k(\overline{A\tau})$ is not contained in any cone of Σ.

Let D be any ample divisor, using the same notation as in the proof of the above lemma, with $B = A^k$, we can see that $a_\tau > b_\tau$, since the $A(u_i)$'s are not in a same cone, and ψ is strictly convex.

Thus the difference between the support functions of $(f_A^{k+1})^* D$ and that of $(f^*)^{k+1} D$ is a nonnegative function which is strictly positive on τ, hence cannot be linear. This means $(f^{k+1})^* D \neq (f^*)^{k+1} D$ in $\mathrm{Pic}(X)$. $\square$

5.3. **Applications of the criterion.** We will apply the above criterion (Theorem 5.2) to give some results about stabilization in certain cases.

First, suppose all entries of A are non-negative, i.e., f_A is a polynomial monomial map. There is a nice nonsingular toric model on which f_A is algebraically stable, namely $(\mathbb{P}^1)^n$.

Proposition 5.6. *Every monomial polynomial map is strongly algebraically stable on $(\mathbb{P}^1)^n$, hence algebraically stable.*

Proof. Let Σ be the fan such that $X(\Sigma) = (\mathbb{P}^1)^n$. The rays of Σ are given by $\tau_i = \mathbb{R}_{\geq 0} \cdot e_i$ and $-\tau_i$, for $i = 1, \cdots, n$. The morphism A maps each of τ_i into the cone σ_+ generated by $e_1, \cdots, e_n$, and maps each of $-\tau_i$ into the cone σ_- generated by $-e_1, \cdots, -e_n$.

Observe that the compositions of polynomial maps are still polynomial maps. So A^k are all polynomial monomial maps for $k \geq 1$. Also notice that $A^k(\tau_i) \subset \sigma_+$, so $\overline{A^k(\tau_i)}$ is a face of σ_+. Hence there is a subset of indexes $I \subset \{1, \cdots, n\}$ such that $\overline{A^k(\tau_i)}$ is generated by $\{e_i | i \in I\}$. Since each $A^k(e_i) \in \sigma_+$, we have that $A(\overline{A^k(\tau_i)}) \subset \sigma_+$. This means $\overline{A^k(\tau_i)}$ maps regularly for all k. By symmetry, we also know that $A(\overline{A^k(-\tau_i)}) \subset \sigma_-$. Therefore, the map f_A is strongly algebraically stable on $X(\Sigma) = (\mathbb{P}^1)^n$. $\square$

We will discuss more properties of monomial maps on $(\mathbb{P}^1)^n$ in Section 7. The above property is about maps on a fixed toric variety $(\mathbb{P}^1)^n$. Next, we will fix some map, and ask whether there exists a toric variety on which the map is strongly algebraically stable. We give partial answers for maps satisfying some conditions.

Theorem 5.7. *Suppose that $A \in \mathbf{M}_n(\mathbb{Z})$ is an integer matrix.*

(1) *If there is a unique eigenvalue λ of A of maximal modulus, with algebraic multiplicity one; then $\lambda \in \mathbb{R}$, and there exists a simplicial toric birational model X (maybe singular) and a $k \in \mathbb{N}$ such that f_A^k is strongly algebraically stable on X.*

(2) *If $\lambda, \bar{\lambda}$ are the only eigenvalues of A of maximal modulus, also with algebraic multiplicity one, and if $\lambda = |\lambda| \cdot e^{2\pi i \theta}$, with $\theta \notin \mathbb{Q}$; then there is no toric birational model which makes f_A strongly algebraically stable.*

Proof. For (1), let $v \in \mathbb{R}^n$ be the eigenvector corresponding to the largest real eigenvalue λ, then the subspace $\mathbb{R}v$ is attracting. We can find integral vectors $v_1, \cdots, v_n$, linearly independent over $\mathbb{R}$, such that

- v is in the interior of the cone generated by $v_1, \cdots, v_n$.
- $A^n(\mathbb{R}v_i) \to \mathbb{R}v$ for all $i = 1, \cdots, n$, as elements of $\mathbb{RP}^n$.

The rays $\{\mathbb{R}_{\geq 0} \cdot v_i, \mathbb{R}_{\geq 0} \cdot (-v_i) \mid i = 1 \cdots, n\}$ generates a fan Σ similar to the way we form $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. That is, the maximal cones of Σ are generated by the sets $\{s_1 v_1, \cdots, s_n v_n\}$ where $s_i \in \{+1, -1\}$. All other cones are faces of some maximal cone. It is easy to see that for some k, f_A^k is strongly algebraically stable on $X(\Sigma)$.

To prove (2), let $\lambda, \bar{\lambda}$ be the largest eigenvalue pair, and $\Gamma \subset \mathbb{R}^n$ be the two dimensional invariant subspace corresponding to them. Since the fan Σ is complete, there is at least one ray $\tau \in \Sigma(1)$ such that under iterations, $A^k \tau$ will approach Γ. Moreover, since $A|_\Gamma$ is an irrational rotation on rays, we know that for all $v \in \Gamma$, there is a sequence k_i such that $A^{k_i} \tau \to \mathbb{R}_{\geq 0} \cdot v$.

Consider the set $\Sigma \cap \Gamma = \{\sigma \cap \Gamma \mid \sigma \in \Sigma\}$, it is a fan in Γ. Each cone in it is strictly convex, but not necessarily rational. Pick $v_0 \in \Gamma$ which lies in the interior of some two dimensional cone of $\Sigma \cap \Gamma$, and pick a sequence k_i such that $A^{k_i} \tau \to \tau_0 = \mathbb{R}_{\geq 0} \cdot v_0$.

Since $A^{k_i} \tau \to \tau_0$ and Σ consists of only finitely many cones, there must be some k such that $\tau_0 \in \overline{A^k \tau}$. But τ_0 is in the interior of some two dimensional cone of $\Sigma \cap \Gamma$, so we know that $\overline{A^k \tau} \cap \Gamma$ is a two dimensional cone in Γ. Finally, we know that $\overline{A^k \tau} \cap \Gamma$ cannot map regularly under all A^k, so $\overline{A^k \tau}$ cannot either. Thus A can never be made strongly algebraically stable. $\square$

We do not know what the correct statement would be for the missing case $\lambda = |\lambda| \cdot e^{2\pi i \theta}$, with $\theta \in \mathbb{Q}$.

Remark. Some of our results in the last section and this section were obtained independently by Mattias Jonsson and Elizabeth Wulcan [9]. They

obtained the pull back formula and the criterion for stability. One of the main theorems in their paper [9, Theorem A'] deal with smooth stabilization of a monomial map by refining a given fan. This aspect of the stabilization is more delicate and is not discussed in our paper. Part (1) of Theorem 5.7 in the current paper is similar to the Theorem B in [9]. The difference is that they have further assumption on the eigenvalues, thus they can guarantee that f_A is already stable. They also discuss the special case of monomial maps on toric surfaces (two dimensional toric varieties), which is not dealt in this paper.

6. Monomial maps on projective spaces

The motivation for studying toric rational maps comes from the study of monomial maps on projective spaces. So let us come back to monomial maps and try to understand more about them with the help of techniques from toric varieties. Some results in this section is well known, but we give another proof from a toric viewpoint.

6.1. **Pulling back divisors and divisor classes.** In this subsection, we will show that pulling-back divisors tells us information about homogenization of a monomial map on projective spaces, and pulling-back divisors classes tells us information about the degree of a monomial map on projective spaces.

Given an $n \times n$ integer matrix $A = (a_{i,j})_{1 \le i,j \le n}$, the associated monomial map $\mathbb{C}^n \to \mathbb{C}^n$ is given by

$$(X_1, \cdots, X_n) \longmapsto \left(\prod_{j=1}^n X_j^{a_{1,j}}, \cdots, \prod_{j=1}^n X^{a_{n,j}} \right).$$

Then we use the embedding $\mathbb{C}^n \hookrightarrow \mathbb{P}^n$ defined by $(X_1, \cdots, X_n) \mapsto [1; X_1; \cdots; X_n]$ to identify $\mathbb{C}^n$ with the open subset $U_0 = \{x_0 \ne 0\} \subset \mathbb{P}^n$. The inverse map $U_0 \to \mathbb{C}^n$ is given by $X_i = x_i/x_0$, and this is used to homogenize the monomial map. After homogenizing, there is another integer matrix, with size $(n+1) \times (n+1)$, denoted by $h(A) = (b_{i,j})_{0 \le i,j \le n}$, such that

$$f_A([x_0; \cdots; x_n]) = \left[\prod_{j=0}^n x_j^{b_{0,j}}; \cdots; \prod_{j=0}^n x_j^{b_{n,j}} \right].$$

Recall the structure of the fan associated to the projective space. The one dimensional cones are generated by the standard basis $e_1, \cdots, e_n$ and $e_0 = -(e_1 + \cdots + e_n)$. Denote them by $\tau_i = \mathbb{R}_{\ge 0} \cdot e_i$ for $i = 0, \cdots, n$. Consider the divisors $D_i = V(\tau_i) = \{x_i = 0\}$. If we want to pull it back by f_A, what we do is to pull back the defining equation. This will give us the equation $\prod_{j=0}^n x_j^{b_{i,j}} = 0$, which means

$$f_A^*(D_i) = b_{i,0} \cdot D_0 + b_{i,1} \cdot D_1 + \cdots + b_{i,n} \cdot D_n.$$

On the other hand, by Theorem 4.1, if ψ_i is the support function of the divisor D_i, then

$$f_A^*(D_i) = -\psi_i(Ae_0) \cdot D_0 - \psi_i(Ae_0) \cdot D_1 - \cdots - \psi_i(Ae_n) \cdot D_n.$$

Thus we obtain the equality $b_{i,j} = -\psi_i(Ae_j)$.

Example 6.1. Consider the monomial map f_A associated to the matrix $A = \begin{pmatrix} -1 & -2 \\ 2 & 0 \end{pmatrix}$. We know $f_A(X,Y) = (X^{-1}Y^{-2}, X^2)$ on $\mathbb{C}^2$, and then using homogenization, we can write down the formula for $f_A : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ as $f_A[w;x;y] = [w^2xy^2; w^5; x^3y^2]$.

Let us consider the divisor $V(\tau_0) = \{w = 0\}$. By pulling back the defining function, we know $f_A^*(V(\tau_0))$ is defined by $w^2xy^2 = 0$, which, as a divisor, is $2 \cdot V(\tau_0) + 1 \cdot V(\tau_1) + 2 \cdot V(\tau_2)$. To apply the above discussion, remember that $V(\tau_0)$ corresponds to the support function ψ_0 on Σ such that $\psi_0(e_0) = -1$ and $\psi_0(e_1) = \psi_0(e_2) = 0$. It is not hard to see that $\psi_0(a_1, a_2) = \min\{0, a_1, a_2\}$. Therefore,

$$\begin{aligned} f_A^*(V(\tau_0)) &= -\psi_0(Ae_0) \cdot V(\tau_0) - \psi_0(Ae_1) \cdot V(\tau_1) - \psi_0(Ae_2) \cdot V(\tau_2) \\ &= 2 \cdot V(\tau_0) + 1 \cdot V(\tau_1) + 2 \cdot V(\tau_2). \end{aligned}$$

In general, the formulae of ψ_i for $\mathbb{P}^n$ is as follows.

$$\begin{cases} \psi_0(a_1, \cdots, a_n) = \min\{0, a_1, \cdots, a_n\}, \\ \psi_i(a_1, \cdots, a_n) = \min\{0, -a_i, a_j - a_i; j \neq i\} \text{ for } i = 1, \cdots, n. \end{cases} \tag{6.1}$$

Since the homogenization matrix $h(A) = (-\psi_i(Ae_j))$ is related to the pulling back divisors, we can translate the condition of algebraic stability to a condition on $h(A)$.

Proposition 6.1. *A monomial map $f_A : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is strongly algebraically stable if and only if $h(A^k) = h(A)^k$ for $k = 1, 2, \cdots$.*

Proof. The entries of the i-th row in $h(A^k)$ are the coefficients of $(f_A^k)^*(V(\tau_i))$, whereas the entries of the i-th row in $h(A)^k$ are the coefficients of $(f_A^*)^k(V(\tau_i))$. The proposition then follows from the fact that $\mathrm{CDiv}_T(\mathbb{P}^n)$ is generated by $V(\tau_0), \cdots, V(\tau_n)$. ☐

What happened in unstable cases is that when we iterate the map $h(A)$ directly, some terms got canceled out. For example, the map we mentioned above $f_A : [w;x;y] \mapsto [w^2xy^2; w^5; x^3y^2]$ is not stable, because when we iterate once, the map

$$[w;x;y] \mapsto [w^9x^8y^8; w^{10}x^5y^{10}; w^{15}x^6y^4]$$

which corresponds to $h(A)^2$, has a common factor $w^9x^5y^4$ on each component; so we need to divide all components by $w^9x^5y^4$, and obtain $[w;x;y] \mapsto [x^3y^4; wy^6; w^6x]$, whose components corresponds to $h(A^2)$.

Next, we turn our attention to the pull back of divisor classes, i.e., elements of $\mathrm{Pic}(\mathbb{P}^n)$. It is well known that $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$, and the isomorphism is given by the degree. We thus have the map $\deg : \mathrm{CDiv}_T(\mathbb{P}^n) \to \mathrm{Pic}(\mathbb{P}^n)$ given by $\deg(\sum a_i V(\tau_i)) = \sum a_i$. Furthermore, for a monomial map f_A on the projective space, it is easy to deduce that the pull back on Picard group is the same as the degree of the map, and is given by $\deg(f_A^* D)$ for any divisor D of degree one. We also denote this number by $\deg(f_A)$. If ψ is the

support function for D, then we know the degree of the monomial map f_A is given by

$$\deg(f_A) = \sum_{i=0}^{n} -\psi(Ae_i). \tag{6.2}$$

For example, let ψ be any one of the ψ_i listed in (6.1), then we can get a concrete formula for $\deg(f_A)$. In particular, let $\psi = \psi_0$, we have

$$\begin{aligned} -\psi(a_1, \cdots, a_n) &= -\psi_0(a_1, \cdots, a_n) \\ &= -\min\{0, a_1, \cdots, a_n\} \\ &= \max\{0, -a_1, \cdots, -a_n\}. \end{aligned}$$

Then we rediscover the formula in [8, Proposition 2.14].

$$\deg(f_A) = \sum_{j=1}^{n} \max_{1\le i\le n}\{0, -a_{ij}\} + \max_{1\le i\le n}\Big\{0, \sum_{j=1}^{n} a_{ij}\Big\}.$$

The definition of algebraic stability for rational maps on $\mathbb{P}^n$ states that f_A is algebraically stable if and only if $\deg(f_A^k) = \deg(f_A)^k$ for all k. Another property of the degree sequence is that it is *submultiplicative*, i.e., $\deg(f_A^{k+k'}) \le \deg(f_A^k) \cdot \deg(f_A^{k'})$. We will use this property several times in the next subsection.

6.2. **Estimates of the degree sequence.** In this subsection, we are going to study the degree sequence $\{\deg(f_A^k)\}_{k=1}^{\infty}$. We are particularly interested in the asymptotic behavior of the degree sequence. An important numerical invariant is the *asymptotic degree growth*

$$\delta_1(f_A) = \lim_{k\to\infty} (\deg(f_A^k))^{\frac{1}{k}}.$$

It is known that for a monomial map f_A, $\delta_1(f_A) = \rho(A)$, the spectral radius of the matrix A ([8, Theorem 6.2]). We will refine this result and give more precise estimates on the degree growth of a monomial map.

Remark. For $\mathbb{P}^n$, the asymptotic degree growth is the same as the first dynamical degree, which will introduce in more detail in Section 7.2.

For two sequences $\{\alpha_k\}_{k=1}^{\infty}$ and $\{\beta_k\}_{k=1}^{\infty}$ of positive real numbers, we say that they are *asymptotically equivalent*, denoted by $\alpha_k \sim \beta_k$, if there exists two positive constants $c_1 \ge c_0 > 0$, independent of k, such that $c_0 \cdot \beta_k \le \alpha_k \le c_1 \cdot \beta_k$ for all k.

Let us start from some examples. In the following examples, we assume that $a > b > 0$ are two natural numbers.

Example 6.2. For $A = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, i.e., f_A is the map $(x, y) \mapsto (x^a, y^b)$ on $\mathbb{C}^2$. It is easy to see that $\deg(f_A^k) = a^k$.

Example 6.3. For $A = \left(\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right)$, then $A^k = \left(\begin{smallmatrix} a^k & ka^{k-1} \\ 0 & a^k \end{smallmatrix}\right)$, i.e., f_A^k is the map $(x, y) \mapsto (x^{a^k} y^{ka^{k-1}}, y^{a^k})$ on $\mathbb{C}^2$. Thus for large k (more precisely, for $k \geq a$),

$$\deg(f_A^k) = a^k + ka^{k-1} = (1 + k/a) \cdot a^k \sim k \cdot a^k.$$

This shows that in the non-diagonalizable case, we may have some polynomial multiplying with the power of the spectral radius in the estimate.

Example 6.4. For $A = \left(\begin{smallmatrix} -a & 0 \\ 0 & b \end{smallmatrix}\right)$, i.e., f_A is the map $(x, y) \mapsto (x^{-a}, y^b)$ on $\mathbb{C}^2$. It is not hard to verify that

$$\deg(f_A^k) = \begin{cases} a^k & \text{for } k \text{ even,} \\ a^k + b^k & \text{for } k \text{ odd.} \end{cases}$$

The degree depends on the parity of k, but we still have $\deg(f_A^k) \sim a^k$. Moreover, the sequence $\{\deg(f_A^k)/a^k\}_{k=1}^{\infty}$ has only one limit point.

Example 6.5. For $A = \left(\begin{smallmatrix} -a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & b \end{smallmatrix}\right)$, the map $f_A : \mathbb{C}^3 \to \mathbb{C}^3$ is given by $(x, y, z) \mapsto (x^{-a}, y^{-a}, z^b)$. The degree sequence of f_A is

$$\deg(f_A^k) = \begin{cases} a^k & \text{for } k \text{ even,} \\ 2 \cdot a^k + b^k & \text{for } k \text{ odd.} \end{cases}$$

We still have $\deg(f_A^k) \sim a^k$, but the sequence $\{\deg(f_A^k)/a^k\}_{k=1}^{\infty}$ has two limit points: 1 and 2.

Example 6.6. Moreover, consider the monomial map $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ defined by $f(x_1, \cdots, x_n) = (x_2^{-a}, x_3^a, \cdots, x_n^a, x_1^a)$. A careful calculation will show that $\deg(f^k) \sim a^k$, and the sequence $\{\deg(f_A^k)/a^k\}_{k=1}^{\infty}$ has n limit points: $1, 2, \cdots, n-1$ and n.

Example 6.7. Consider the monomial map f_A associated to the matrix $A = \left(\begin{smallmatrix} -1 & -2 \\ 2 & 0 \end{smallmatrix}\right)$, as in Example 6.1. The matrix A has two conjugate eigenvalues $\lambda = (-1 + \sqrt{-15})/2$ and $\bar{\lambda}$; and $\lambda/\bar{\lambda}$ is not a root of unity. We will show in Theorem 6.2 that $\deg(f_A^k) \sim |\lambda|^k$, but when we consider the sequence $\{\deg(f_A^k)/|\lambda|^k\}_{k=1}^{\infty}$, we no longer have finitely many limit points. In fact, we will prove that the sequence is dense in some interval (Proposition 6.9).

The main result for general monomial maps is the following theorem.

Theorem 6.2. *Given an $n \times n$ integer matrix A with nonzero determinant, assume that $\rho(A)$ is the spectral radius of A. Then there exist two positive constants $C_1 \geq C_0 > 0$ and a unique integer ℓ with $0 \leq \ell \leq n-1$, such that*

$$C_0 \cdot k^{\ell} \cdot \rho(A)^k \leq \deg(f_A^k) \leq C_1 \cdot k^{\ell} \cdot \rho(A)^k \tag{6.3}$$

for all $k \in \mathbb{N}$. Or, equivalently, $\deg(f_A^k) \sim k^{\ell} \cdot \rho(A)^k$.

In fact, $(\ell + 1)$ is the size of the largest Jordan block of A among the ones corresponding to eigenvalues of maximal modulus.

The idea we use to prove the theorem is the following observation. The assignment $A \mapsto \deg(f_A)$ can be extended naturally to a function $\nu : \mathbf{M}_n(\mathbb{R}) \to \mathbb{R}$, and the function ν is *almost* a norm on $\mathbf{M}_n(\mathbb{R})$. Thus some techniques on norms also applies to the study of degrees.

More precisely, in formula (6.2), notice that the right hand side can be defined over the real numbers because ψ is a continuous piecewise linear function defined on $N_{\mathbb{R}} \cong \mathbb{R}^n$. The only requirement is that the associated divisor of ψ has degree one. Thus, we define a function $\nu : \mathbf{M}_n(\mathbb{R}) \to \mathbb{R}$ by

$$\nu(M) = \sum_{i=0}^{n} -\psi(Me_i). \tag{6.4}$$

Proposition 6.3. *The following properties hold for the function ν.*

(i) *Any support function ψ of a T-divisor of degree one on $\mathbb{P}^n$ will give the same ν, i.e., ν is independent of the choice of ψ.*
(ii) *ν is a continuous function when we equip $\mathbf{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ and $\mathbb{R}$ with the usual topology of the Euclidean spaces.*
(iii) *$\nu(M) \geq 0$, and $\nu(M) = 0$ if and only if $M = 0$. Thus, in fact, we have $\nu : \mathbf{M}_n(\mathbb{R}) \to \mathbb{R}_{\geq 0}$.*
(iv) *$\nu(rM) = r \cdot \nu(M)$ for $r \geq 0$.*
(v) *$\nu(M + M') \leq \nu(M) + \nu(M')$.*

Proof. First, notice that (ii) is true because ψ is continuous, and (iv) is true because ψ is linear on each ray. Then (i) follows by (ii), (iv), and the fact that $\nu(A) = \deg(f_A)$ for $A \in \mathbf{M}_n(\mathbb{Z})$, which is independent of ψ.

Once we know that ν is independent of the choice of ψ, one can pick any ψ, e.g. $\psi = \psi_0$, and prove (iii) and (v) directly. However, we would like to offer a more intrinsic explanation for (iii) and (v).

Since ψ is the support function for a degree one divisor D on $\mathbb{P}^n$, we know that D is very ample, and hence ψ is strictly convex (see [6, p.70]). The first part of (iii), and (v), can be easily deduced from convexity. Strict convexity is needed to show that $\nu(M) = 0$ implies $M = 0$.

Suppose $M \neq 0$, then $Me_0, Me_1, \cdots, Me_n$ cannot be all zero. But since $Me_0 + \cdots + Me_n = 0$, and the cones in the fan for $\mathbb{P}^n$ are strongly convex (they do not contain any line through the origin), $Me_0, \cdots, Me_n$ cannot all lie in the same cone. Thus by strict convexity, we know

$$\nu(M) = -\sum_{i=0}^{n} \psi(Me_i) > -\psi(\sum_{i=0}^{n} Me_i) = \psi(0) = 0.$$

□

By properties (iii)–(v), we know that the only reason to prevent ν from being a norm is that we may have $\nu(M) \neq \nu(-M)$. Indeed, for the $n \times n$ identity matrix I_n, we have $\nu(I_n) = 1$, but $\nu(-I_n) = n - 1$. So ν is not a norm. However, if we define $\bar{\nu}(M) = \nu(M) + \nu(-M)$, then $\bar{\nu}$ is a norm.

Before we prove Theorem 6.2, we need an elementary lemma from linear algebra.

Lemma 6.4. *For an $n \times n$ matrix $A \in \mathbf{M}_n(\mathbb{C})$ and any norm $\|\cdot\|$ defined on $\mathbf{M}_n(\mathbb{C})$, there exists two positive constants $c_1 \geq c_0 > 0$ and a unique integer ℓ with $0 \leq \ell \leq n-1$, such that*

$$c_0 \cdot k^\ell \cdot \rho(A)^k \leq \|A^k\| \leq c_1 \cdot k^\ell \cdot \rho(A)^k \tag{6.5}$$

for all $k \in \mathbb{N}$. Here $\rho(A)$ is the spectral radius of A, and $(\ell+1)$ is the size of the largest Jordan block among those blocks corresponding to eigenvalues of maximal modulus $\rho(A)$.

Proof. It suffices to prove the lemma for the L^∞ norm on $\mathbf{M}_n(\mathbb{C})$. Thus, for $A = (a_{ij})$, we set $\|A\| = \|A\|_\infty = \max_{i,j}\{|a_{ij}|\}$ for the rest of the proof.

Observe that $\|AB\| \leq n \cdot \|A\| \cdot \|B\|$. If we write $A = PJP^{-1}$, where J is the Jordan canonical form of A, then we have

$$(n^2 \cdot \|P\| \cdot \|P^{-1}\|)^{-1} \cdot \|J^k\| \leq \|A^k\| \leq (n^2 \cdot \|P\| \cdot \|P^{-1}\|) \cdot \|J^k\|.$$

For large k, it is easy to see that $\|J^k\| = \binom{k}{\ell} \cdot \rho(A)^k \sim k^\ell \cdot \rho(A)^k$, where $(\ell+1)$ is as described above. Hence the lemma follows. □

Now we are ready to proof the theorem.

Proof of Theorem 6.2. For the matrix $A \in \mathbf{M}_n(\mathbb{Z})$, consider the set

$$\left\{ \frac{A^k}{k^\ell \rho(A)^k} \;\middle|\; k \in \mathbb{N} \right\} \subset \mathbf{M}_n(\mathbb{R}).$$

By lemma 6.4, it is a subset of a compact set $S = \{M \in \mathbf{M}_n(\mathbb{R}) \mid c_0 \leq \|M\| \leq c_1\}$ for some $c_1 \geq c_0 > 0$. Since ν is continuous, we have $\nu(S) \subset [C_0, C_1]$ for some reals $C_1 \geq C_0 \geq 0$. Moreover, $0 \notin S$, thus $C_0 > 0$. This gives us

$$C_0 \leq \nu\Big(\frac{A^k}{k^\ell \cdot \rho(A)^k}\Big) \leq C_1.$$

for all $k \in \mathbb{N}$, with $C_1 \geq C_0 > 0$.

Finally, since $k^\ell \cdot \rho(A)^k > 0$, and $\nu(A^k) = \deg(f_A^k)$, we have

$$C_0 \cdot k^\ell \cdot \rho(A)^k \leq \deg(f_A^k) \leq C_1 \cdot k^\ell \cdot \rho(A)^k$$

This concludes the proof. □

Corollary 6.5. *If A is diagonalizable, then we have*

$$C_0 \cdot \rho(A)^k \leq \deg(f_A^k) \leq C_1 \cdot \rho(A)^k \tag{6.6}$$

for some constants $C_1 \geq C_0 \geq 1$.

Proof. In the diagonalizable case, $\ell = 0$, hence we have (6.6). Recall that the degree sequence is submultiplicative. Thus, if we have $\frac{\deg(f_A^k)}{\rho(A)^k} = r < 1$ for some k, then

$$\frac{\deg(f_A^{kj})}{\rho(A)^{kj}} \leq \frac{\deg(f_A^k)^j}{\rho(A)^{kj}} = r^j \to 0 \text{ as } j \to +\infty.$$

This contradicts the existence of $C_0 > 0$. Therefore, $\frac{\deg(f_A^k)}{\rho(A)^k} \geq 1$ for all k, and we can choose $C_0 \geq 1$. □

If we impose more conditions on the matrix A, we can obtain more precise estimates on the degree sequence.

Theorem 6.6. *Assuming that the matrix A is diagonalizable, and there is a unique eigenvalue λ_1 of maximal modulus, which is real and positive. Also, assume that the eigenvalues of A are arranged as $\lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_m|$ for some m. Then there is a constant $C \geq 1$ such that*

$$\deg(f_A^k) = C \cdot \lambda_1^k + O(|\lambda_2|^k).$$

Proof. First, given a vector $v \in \mathbb{R}^n$, since A is diagonalizable, we can represent v uniquely as

$$v = v_1 + v_2 + \cdots + v_m,$$

where each $v_j \in \mathbb{C}^n$ is an eigenvector corresponding to λ_j. We have $v_1 \in \mathbb{R}^n$ since λ_1 is real. Thus

$$A^k v = \lambda_1^k v_1 + \lambda_2^k v_2 + \cdots + \lambda_m^k v_m.$$

Let ψ be the support function of some degree one divisor D in $\mathbb{P}^n$. For each k, there is some maximal cone σ_k such that $A^k v \in \sigma_k$. Let L_k be the linear function such that $L_k|_{\sigma_k} = \psi|_{\sigma_k}$. Notice that L_k can be defined on $\mathbb{C}^n$ as a linear map, and we have

$$\begin{aligned}
\psi(A^k v) = L_k(A^k v) &= L_k\big(\sum_{j=1}^m \lambda_j^k v_j\big) \\
&= \lambda_1^k \cdot L_k(v_1) + \sum_{j=2}^m \lambda_j^k \cdot L_k(v_j) \\
&= L_k(v_1) \cdot \lambda_1^k + O(|\lambda_2|^k).
\end{aligned}$$

There are two cases here: $v_1 \neq 0$, or $v_1 = 0$.

First, if $v_1 \neq 0$, then for the rays $\tau = \mathbb{R}_{\geq 0} \cdot v$, we know $A^k \tau \to \mathbb{R}_{\geq 0} \cdot v_1$. Thus for large k, we can choose σ_k so that both $A^k v \in \sigma_k$ and $v_1 \in \sigma_k$. Since $L_k|_{\sigma_k} = \psi|_{\sigma_k}$ for the cone σ_k, we know that for large k, the value $L_k(v_1) = \psi(v_1)$ is independent of k, and $\psi(A^k v) = \psi(v_1) \cdot \lambda_1^k + O(|\lambda_2|^k)$. Second, if $v_1 = 0$, then it is obvious that $\psi(A^k v) = O(|\lambda_2|^k)$.

Now let's look at the fan structure of projective spaces. For the ray generators $e_0, e_1, \cdots, e_n$ of $\mathbb{P}^n$, if e_i is decomposed as

$$e_i = v_{i,1} + v_{i,2} + \cdots + v_{i,m} \tag{6.7}$$

for $i = 0, 1, \cdots, n$, where each $v_{i,j}$ is an eigenvector corresponding to the eigenvalue λ_j. Then

$$\psi(A^k e_i) = \psi(v_{i,1}) \cdot \lambda_1^k + O(|\lambda_2|^k).$$

If we set

$$C = \sum_{i=0}^{n} -\psi(v_{i,1}), \tag{6.8}$$

then we can compute the degree sequence $\deg(f_A^k)$ as

$$\begin{aligned}
\deg(f_A^k) &= \deg((f_A^k)^* D) \\
&= \sum_{i=1}^{n} -\psi(A^k e_i) \\
&= C \cdot \lambda_1^k + O(|\lambda_2|^k).
\end{aligned}$$

The fact that $C \geq 1$ is a consequence of Corollary 6.5. □

Notice that, on our way to prove the theorem, we also derive a concrete formula for the constant C in (6.8).

Theorem 6.7. *Assuming that the matrix A is diagonalizable, and there is a unique eigenvalue λ_1 of maximal modulus, which is real and negative. Also assume that the eigenvalues of A are arranged as $(-\lambda_1) > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_m|$ for some m. Then there are two positive constants C_0, C_1, not necessarily distinct, and satisfying $1 \leq C_0 \leq C_1^2$, such that*

$$\deg(f_A^{2k+l}) = C_l \cdot |\lambda_1|^{2k+l} + O(|\lambda_2|^{2k+l}),$$

where $l = 0, 1$.

Proof. We consider the subsequences $\{\deg(f_A^{2k})\}$ and $\{\deg(f_A^{2k+1})\}$. Since A^2 satisfies the condition in Theorem 6.6, with the unique eigenvalue $|\lambda_1|^2$ with maximal modulus, thus

$$\deg(f_A^{2k}) = C_0 \cdot |\lambda_1|^{2k} + O(|\lambda_2|^{2k})$$

for some $C_0 \geq 1$. For the subsequence $\{\deg(f_A^{2k+1})\}$, we consider Ae_i instead of e_i in (6.7) in the proof of Theorem 6.6, and apply the map f_A^2 on these vectors. We then get

$$\deg(f_A^{2k+1}) = C_1 \cdot |\lambda_1|^{2k+1} + O(|\lambda_2|^{2k+1})$$

for some $C_1 \geq 1$.

Finally, for any k, we have,

$$\deg(f_A^{4k+2})/|\lambda_1|^{4k+2} \leq \big(\deg(f_A^{2k+1})/|\lambda_1|^{2k+1}\big)^2.$$

As $k \to \infty$, the left side converges to C_0, while the right side converges to C_1^2. So the relation $C_0 \leq C_1^2$ follows. This completes the proof. □

From the proof, we cannot tell if the two constants C_0 and C_1 are the same or not. Notice that Examples 6.4 and 6.5 are both examples of the theorem. However, we have $C_0 = C_1 = 1$ for Example 6.4, but $C_0 = 1$, $C_1 = 2$ for Example 6.5. This shows that both cases are possible.

The idea in the proof of Theorem 6.7 of considering subsequences can be pushed further to prove the following more general result.

Theorem 6.8. *Assume that the matrix A is diagonalizable, and assume for each eigenvalue λ of A of maximum modulus, $\lambda/\bar{\lambda}$ is a root of unity. Then there is a positive integer p, and p constants $C_0, C_1, \cdots, C_{p-1} \geq 1$, such that*

$$\deg(f_A^{pk+l}) = C_l \cdot |\lambda_1|^{pk+l} + O(|\lambda_2|^{pk+l}),$$

where $l = 0, 1, \cdots, p-1$.

Proof. Notice that there is an integer p such that the eigenvalue of A^p of maximum modulus is unique and positive, so we can use the same argument as Theorem 6.7 to the subsequences

$$\{\deg(f_A^{pk})\}, \{\deg(f_A^{pk+1})\}, \cdots, \{\deg(f_A^{pk+p-1})\}.$$

The theorem then follows. □

Under the assumption of Theorem 6.8, the sequence $\{\deg(f_A^k)/|\lambda_1|^k\}_{k=1}^{\infty}$ has finitely many limit points, namely, $C_0, \cdots, C_{p-1}$. The following proposition shows a different behavior of the sequence $\{\deg(f_A^k)/|\lambda_1|^k\}_{k=1}^{\infty}$ when we have a maximal eigenvalue λ such that $\lambda/\bar{\lambda}$ is *not* a root of unity. Therefore, we cannot expect Theorem 6.8 holds for general diagonalizable matrices.

Proposition 6.9. *For a 2×2 integer matrix A, suppose it has a conjugate pair $\lambda, \bar{\lambda}$ of eigenvalues such that $\lambda/\bar{\lambda}$ is not a root of unity. Then the sequence $\{\deg(f_A^k)/|\lambda|^k\}_{k=1}^{\infty}$ is dense in some closed interval contained in $[1, \infty)$.*

Proof. First, notice that

$$\frac{\deg(f_A^k)}{|\lambda|^k} = \frac{\nu(A^k)}{|\lambda|^k} = \nu\Big(\frac{A^k}{|\lambda|^k}\Big) = \nu\Big((A/|\lambda|)^k\Big). \tag{6.9}$$

Since $\lambda/\bar{\lambda}$ is not a root of unity, we can conjugate $A/|\lambda|$ to some irrational rotation matrix, i.e., we can write

$$A/|\lambda| = P \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot P^{-1},$$

for some $\theta \notin 2\pi\mathbb{Q}$. Thus the closure of the set $S = \{(A/|\lambda|)^k | k \in \mathbb{N}\}$ is

$$\overline{S} = \Big\{ P \cdot \big(\begin{smallmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{smallmatrix}\big) \cdot P^{-1} \;\Big|\; t \in [0, 2\pi] \Big\}.$$

$\overline{S}$ is, topologically, a circle inside $\mathbf{M}_2(\mathbb{R})$. Since ν is continuous, $\nu(\bar{S}) = \overline{\nu(S)}$ is connected and compact. Thus it is either a point or a closed interval.

We claim that $\overline{\nu(S)}$ cannot be a point. If $\overline{\nu(S)} = \{C\}$, then we will have $\deg(f_A^k) = C \cdot |\lambda|^k$ for all $k \in \mathbb{N}$. In this case, the degree sequence $d_k = \deg(f_A^k)$ satisfies a linear recurrence $d_{k+1} = |\lambda| \cdot d_k$. This contradicts a theorem of Bedford and Kim [1, Theorem 1.1], which asserts that if the matrix A has a complex eigenvalue λ of maximal modulus, and $\lambda/\bar{\lambda}$ is not a root of unity, then the degree sequence for f_A cannot satisfy any linear recurrence relation.

Hence, $\nu(\bar{S}) = \overline{\nu(S)}$ is a closed interval. By (6.9), $\nu(S)$ is exactly the set $\{\deg(f_A^k)/|\lambda|^k; k \in \mathbb{N}\}$. Finally, by Corollary 6.5, we further know that the interval $\overline{\nu(S)}$ is contained in $[1, +\infty)$. This concludes the proof. □

6.3. **Degree growth on weighted projective spaces.** Weighted projective spaces are generalizations of the usual projective spaces. The results we obtained in the last subsection about the degree growth of monomial maps on projective spaces can be generalized to weighted projective spaces. We will explain briefly how the generalization is done in this section.

For arbitrary positive integers $d_0, \cdots, d_n$, the associated *weighted projective space*, denoted by $\mathbb{P}(d_0, \cdots, d_n)$, is defined as

$$\mathbb{P}(d_0, \cdots, d_n) = (\mathbb{C}^{n+1} - \{0\})/\sim$$

where the equivalent relation is given by $(x_0, \cdots, x_n) \sim (\zeta^{d_0} x_0, \cdots, \zeta^{d_n} x_n)$ for $\zeta \in \mathbb{C}^*$.

For $d = \gcd(d_0, \cdots, d_n)$, one can show that $\mathbb{P}(d_0, \cdots, d_n) \cong \mathbb{P}(d_0/d, \cdots, d_n/d)$ ([10, Proposition 3.6 (I)]). Moreover, suppose that $d_0, \cdots, d_n$ have no common factor, and that d is a common factor of all d_i for $i \neq j$ (and therefore d is coprime to d_j). Then

$$\mathbb{P}(d_0, \cdots, d_n) \cong \mathbb{P}\Big(\frac{d_0}{d}, \cdots, \frac{d_{j-1}}{d}, d_j, \frac{d_{j+1}}{d}, \cdots, \frac{d_n}{d}\Big),$$

(see [10, Proposition 3.6 (II)]). A weighted projective space $\mathbb{P}(d_0, \cdots, d_n)$ such that no n of the $d_0, d_1, \cdots, d_n$ have a common factor is called *well formed*. The above isomorphism allows us only consider the weighted projective spaces which are well formed. We will make that assumption from now on. Also, for simplicity of notation, we will denote $\mathbb{P}(d_0, \cdots, d_n)$ simply by $\mathbb{P}$ when there is no confusion. The usual projective space, which is $\mathbb{P}(1, 1, \cdots, 1)$, will still be denoted by $\mathbb{P}^n$.

To construct $\mathbb{P}(d_0, \cdots, d_n)$ as a toric variety, one uses the same fan as in the construction of the projective spaces. That is, the cones are generated by proper subsets of $\{e_0, \cdots, e_n\}$. The lattice N' is taken to be generated by the vectors $e_i' := e_i/d_i$, $i = 0, \cdots, n$. Let $\tau_i = \mathbb{R}_{\geq 0} \cdot e_i$ be the rays for the fan of $\mathbb{P}$, the well formed-ness of $\mathbb{P}$ implies that e_i' is the ray generator for τ_i for $i = 0, \cdots, n$.

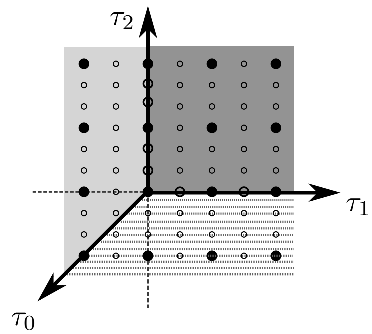

FIGURE 3. The lattice and fan structure for $\mathbb{P} = \mathbb{P}(1, 2, 3)$.

Example 6.8. The lattice and fan structure of $\mathbb{P}(1, 2, 3)$ is shown in Figure 6.8. The solid black dots represent the lattice $N = \mathbb{Z}^2$. The unfilled dots, together with the solid dots, represent the lattice $N' = \frac{1}{2}\mathbb{Z} \oplus \frac{1}{3}\mathbb{Z}$ for $\mathbb{P}(1, 2, 3)$. The fan structure of $\mathbb{P}(1, 2, 3)$ is the same as $\mathbb{P}^2$.

If we define the map $\theta : \mathbb{Z}^{n+1} \to N'$ by $\theta(a_0, \cdots, a_n) = a_0 e'_0 + \cdots + a_n e'_n$, then θ is a surjective homomorphism, and $\ker(\theta)$ is the rank one subgroup $\mathbb{Z} \cdot (d_0, \cdots, d_n)$. Hence $N' \cong \mathbb{Z}^{n+1}/\mathbb{Z} \cdot (d_0, \cdots, d_n)$, and we also obtain a description for the dual lattice as follows:

$$M' = (N')^\vee \cong \big\{(a_0, \cdots, a_n) \in \mathbb{Z}^{n+1} \mid a_0 d_0 + \cdots + a_n d_n = 0\big\}.$$

Recall that the group of T-invariant Weil divisors is freely generated by the orbit closures $V(\tau_i)$, i.e., $\mathrm{WDiv}_T(\mathbb{P}) \cong \oplus_{i=0}^n \mathbb{Z} \cdot V(\tau_i)$. Define the *weighted degree homomorphism* $\deg' : \mathrm{WDiv}_T(\mathbb{P}) \to \mathbb{Z}$ by $\deg'(a_i \cdot V(\tau_i)) = \sum_{i=0}^n a_i d_i$. The map is surjective since $\gcd(d_0, \cdots, d_n) = 1$. Moreover, it is easy to see that the kernel is canonically isomorphic to M'. Therefore, the divisor class group $A_{n-1}(\mathbb{P}) \cong \mathbb{Z}$, and the isomorphism is induced by the weighted degree.

Let $m = \mathrm{lcm}(d_0, \cdots, d_n)$, one can show that a T_N-invariant Weil divisor D is Cartier if and only if $m | \deg(D)$. As a consequence, the image of the Picard group $\mathrm{Pic}(\mathbb{P}) \subset A_{n-1}(\mathbb{P}) \cong \mathbb{Z}$ under the isomorphism is the subgroup $m\mathbb{Z}$.

Therefore, after tensoring with the group of rational numbers $\mathbb{Q}$, the group of T-invariant $\mathbb{Q}$-Weil divisors is the same as that of $\mathbb{Q}$-Cartier divisors, and the divisor class group is the same as the Picard group, both with $\mathbb{Q}$-coefficients. Thus we will look at $\mathbb{Q}$-Weil divisors and rational support function in this subsection.

Let ψ be a rational support function, then ψ induces a $\mathbb{Q}$-Cartier divisor on $\mathbb{P}$, whose associated $\mathbb{Q}$-Weil divisor is $D' = \sum_{i=0}^n -\psi(e'_i) \cdot V(\tau_i)$. Also, ψ induces a $\mathbb{Q}$-Cartier divisor on $\mathbb{P}^n$, with associated $\mathbb{Q}$-Weil divisor $D = \sum_{i=0}^n -\psi(e_i) \cdot V(\tau_i)$. A basic fact is the following.

Lemma 6.10. *Assume the above notations, then the weighted degree of D' is the same as the degree of D, i.e., $\deg'(D') = \deg(D)$.*

Proof. This can be verified as follows:

$$\deg'(D') = \sum_{i=0}^{n} -d_i \cdot \psi(e_i') = \sum_{i=0}^{n} -d_i \cdot \psi(e_i/d_i) = \sum_{i=0}^{n} -\psi(e_i) = \deg(D).$$

□

We can then discuss the pull back map for a toric map on a weighted projective space. The pull back map on divisors can be obtained by the formula in Theorem 4.1. The pull back map on Picard group is given by the action on the degree of a divisor, which we also call it the *weighted degree* of the map, denoted by $\deg'(f_A)$. In general, the weighted degree is a rational number, not necessarily an integer.

More precisely, let $A \in \mathrm{End}(N')$, then A induces a toric rational map $f_A : \mathbb{P} \to \mathbb{P}$. Using the standard basis $e_1, \cdots, e_n$ of $N'_{\mathbb{R}} \cong N_{\mathbb{R}}$, we can represent A as an $n \times n$ matrix with *rational* entries. The following proposition tells us how to compute the weighted degree of f_A.

Proposition 6.11. *Assume the above notations, then the weighted degree of f_A is given by*

$$\deg'(f_A) = \nu(A),$$

where $\nu : \mathbf{M}_n(\mathbb{R}) \to \mathbb{R}$ is the function defined in (6.4).

Proof. The weighted degree can be computed as $\deg'(f_A) = \deg(f_A^* D')$ for any $\mathbb{Q}$-divisor D' on $\mathbb{P}$ of degree one. Thus, if D' is a $\mathbb{Q}$-divisor on $\mathbb{P}$ of degree one, and $\psi = \psi_{D'}$ is the $\mathbb{Q}$-support function of D', then

$$\deg'(f_A) = \sum_{i=0}^{n} -d_i \cdot \psi(Ae_i') = \sum_{i=0}^{n} -\psi(Ae_i) = \nu(A).$$

The last equality holds because the degree of the $\mathbb{Q}$-divisor on $\mathbb{P}^n$ associated to ψ also has degree one by Lemma 6.10. □

Example 6.9. For the matrix $A = \begin{pmatrix} 1 & -\frac{3}{2} \\ \frac{2}{3} & 0 \end{pmatrix}$, it does not preserve the lattice $\mathbb{Z}^2$, but it preserve the lattice $N' = \frac{1}{2}\mathbb{Z} \oplus \frac{1}{3}\mathbb{Z}$ for $\mathbb{P}(1,2,3)$ in Example 6.8. In fact, we have $e_1' \mapsto e_1' + e_2'$ and $e_2' \mapsto -e_1'$. Therefore, f_A is a well-defined toric rational map on $\mathbb{P}(1,2,3)$. The map f_A has degree $\frac{13}{6}$, which is not an integer.

Since the weighted degree function is the same as the function ν, this tells us that the weighted degree growth of iterations of toric rational maps on $\mathbb{P}$ follows the same results as the degree growth on $\mathbb{P}^n$. Therefore, we can conclude that all theorems and propositions in Section 6.2 also hold for weighted degree growth of toric rational maps on weighted projective spaces.

7. Monomial maps and polynomial maps on $(\mathbb{P}^1)^n$

In this section, we fix the space $(\mathbb{P}^1)^n = \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{n}$, and fix the fan Σ to be the one associated to $(\mathbb{P}^1)^n$, as described in Example 2.5. We set the coordinate as

$$(\mathbb{P}^1)^n = \left\{([x_1;y_1],\cdots,[x_n;y_n]) \;\middle|\; [x_i;y_i] \in \mathbb{P}^1 \text{ for } i=1,\cdots,n\right\}.$$

7.1. **Pulling back divisors by monomial maps on** $(\mathbb{P}^1)^n$**.** Let D_i, E_i be the divisors defined by the equations $x_i = 0$, $y_i = 0$, respectively, for $i = 1,\cdots,n$. Notice that if we set $\tau_i = \mathbb{R}_{\geq 0} \cdot e_i$, then $D_i = V(\tau_i)$ and $E_i = V(-\tau_i)$. The group of torus invariant divisors are then generated by the D_i's and E_i's, that is,

$$\mathrm{CDiv}_T((\mathbb{P}^1)^n) \cong \mathbb{Z}^{2n} = \mathbb{Z}D_1 \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}D_n \oplus \mathbb{Z}E_n.$$

Notice that the support function for D_i and E_i are

$$\psi_{D_i}(a_1,\cdots,a_n) = \min\{-a_i, 0\},$$
$$\psi_{E_i}(a_1,\cdots,a_n) = \min\{a_i, 0\}.$$

Consider the monomial map f_A on $(\mathbb{P}^1)^n$ associated to a matrix $A = (a_{ij}) \in \mathbf{M}_n(\mathbb{Z})$. Using Theorem 4.1 and the formula for ψ_{D_i} and ψ_{E_i}, we obtain the following explicit description of the pull back f_A^* on the group $\mathrm{CDiv}_T((\mathbb{P}^1)^n)$. First, fix the ordered basis $D_1, E_1, \cdots, D_n, E_n$. With respect to this basis, f_A^* is represented by a $2n \times 2n$ integer matrix $(\alpha_{ij})_{i,j=1\cdots,n}$. Here each α_{ij} is a 2×2 block:

$$\alpha_{ij} = \begin{cases} \begin{pmatrix} |a_{ji}| & 0 \\ 0 & |a_{ji}| \end{pmatrix} & \text{if } a_{ji} \geq 0, \\ \\ \begin{pmatrix} 0 & |a_{ji}| \\ |a_{ji}| & 0 \end{pmatrix} & \text{if } a_{ji} \leq 0. \end{cases} \tag{7.1}$$

Notice that the block α_{ij} corresponds to the entry a_{ji}, not a_{ij}. This is because when we induce the map f_A from A, the matrix A acts on the lattice N, and we have to take its transposition ${}^tA$ to act on $M = N^\vee$, see Examples 2.1 and 2.2.

Passing to the Picard groups, D_i is linearly equivalent to E_i, so

$$\mathrm{Pic}((\mathbb{P}^1)^n) \cong \mathbb{Z}^n = \mathbb{Z} \cdot [D_1] \oplus \cdots \oplus \mathbb{Z} \cdot [D_n].$$

The pull-back f_A^* on Picard group, with respect to the basis $[D_1],\cdots,[D_n]$, is given by the matrix $\big(|a_{ji}|\big)$, i.e., each entry of f_A^* is the absolute value of the corresponding entry of the transpose matrix ${}^tA$. This observation about f^* has an immediate application, which is shown in the next subsection.

7.2. **The first dynamical degree of a monomial map.** It is known that the first dynamical degree of a monomial map f_A is the spectral radius of the matrix A (see [8, Theorem 6.2]). In this section we will provide an alternative proof of this fact.

For a compact Kähler manifold X, and a rational self map $f : X \to X$, let $f^* : H^{1,1}(X;\mathbb{R}) \to H^{1,1}(X;\mathbb{R})$ be the pull back map on the $(1,1)$ cohomology group. Put any norm $\|\cdot\|$ on the space $\mathrm{End}_{\mathbb{R}}(H^{1,1}(X;\mathbb{R}))$ of $\mathbb{R}$-linear endomorphism on $H^{1,1}(X;\mathbb{R})$. Then the first dynamical degree of f, denoted by $\delta_1(f)$, is defined by

$$\delta_1(f) = \liminf_{k\to\infty} \|(f^k)^*\|^{1/k}.$$

A property of the first dynamical degree is that it is invariant under birational conjugate (see [7, Proposition 2.6 and Corollaire 2.7]). Therefore, it is a common method to find a good birational model $\tilde{X}$ of X so that it is easier to compute the dynamical degree for the conjugate $\tilde{f} : \tilde{X} \to \tilde{X}$. In this section, we would like to use the model $(\mathbb{P}^1)^n$. With this model, we can obtain another proof of the following:

Theorem 7.1. *The first dynamical degree of a monomial map f_A is the spectral radius of the matrix A.*

Proof. For $X = (\mathbb{P}^1)^n$, we have $H^{1,1}(X;\mathbb{R}) \cong \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Also, remember that f_A^* can be represented by the matrix $\big(|a_{ji}|\big)$. For a linear map in $\mathrm{End}_{\mathbb{R}}(H^{1,1}(X;\mathbb{R}))$, we first represent it by a matrix with respect to the ordered basis $[D_1], \cdots, [D_n]$. Then we take the L^1 norm of the matrix representation. This gives a norm on $\mathrm{End}_{\mathbb{R}}(H^{1,1}(X;\mathbb{R}))$, i.e., we set

$$\|f_A^*\| = \|\big(|a_{ji}|\big)\|_1 = \sum_{i,j=1}^{n} |a_{ji}|.$$

It is easy to see that

$$\|f_A^*\| = \|A\|_1. \tag{7.2}$$

Furthermore, for the k-th iterate of f_A, we know $f_A^k = f_{A^k}$. By substituting A with A^k in (7.2), we also know $\|(f_A^k)^*\| = \|(f_{A^k})^*\| = \|A^k\|_1$. Therefore,

$$\begin{aligned}\delta_1(f) &= \liminf_{k\to\infty} \|(f_A^k)^*\|^{1/k} \\ &= \liminf_{k\to\infty} \|A^k\|_1^{1/k} \\ &= \rho(A).\end{aligned}$$

□

7.3. **Algebraic stability of monomial maps on $(\mathbb{P}^1)^n$.** We now turn to the discussion about algebraic stability. The goal of this section is to show that for monomial maps on $(\mathbb{P}^1)^n$, being algebraic stable is equivalent to being strongly algebraic stable.

Let $A=(a_{ij}), B=(b_{ij})$ be two $n\times n$ integer matrices, and f_A, f_B be the monomial maps on $(\mathbb{P}^1)^n$ induced by A, B, respectively. Also, let $C=AB$ be their product, and assume that $C=(c_{ij})$. Then $f_A\circ f_B=f_{AB}=f_C$, thus we know on the Picard group, the pull back $(f_A\circ f_B)^*=f_C^*$ is represented by the matrix whose (i,j)-th entry is given by $|c_{ji}|=|\sum_{k=1}^n a_{jk}b_{ki}|$. On the other hand, the (i,j)-th entry of the matrix representing $f_B^*\circ f_A^*$ is

$$\sum_{k=1}^n |b_{ki}|\cdot|a_{jk}|=\sum_{k=1}^n |a_{jk}b_{ki}|.$$

The two numbers are equal if and only if all the non-zero summands $a_{jk}b_{ki}$ have the same sign, i.e., they are either all positive or all negative. Thus we know that $(f_A\circ f_B)^*=f_B^*\circ f_A^*$ for divisor classes in the Picard group if and only if the following condition holds:

($\star$) For each entry $\sum_{k=1}^n a_{jk}b_{ki}$ of the matrix AB, all the non-zero summands $a_{jk}b_{ki}$ have the same sign.

The situation in the group $\mathrm{CDiv}_T((\mathbb{P}^1)^n)$ is similar, but more complicated. We know that for the fixed ordered basis $D_1, E_1, \cdots, D_n, E_n$, the pull back maps f_A^*, f_B^*, and $(f_A\circ f_B)^*=f_C^*$ on $\mathrm{CDiv}_T(X)$ are represented by the $2n\times 2n$ integer matrices (α_{ij}), (β_{ij}), and (γ_{ij}), respectively. Here each of the α_{ij}, β_{ij}, and γ_{ij} is a 2×2 block as described in (7.1). The composition $f_B^*\circ f_A^*$ is then represented by the matrix product $(\beta_{ij})\cdot(\alpha_{ij})$. We can do the matrix multiplication by blocks, and assume $(\gamma'_{ij})=(\beta_{ij})\cdot(\alpha_{ij})$, where

$$\gamma'_{ij}=\sum_{k=1}^n \beta_{ik}\cdot\alpha_{kj}.$$

A simple calculation shows that

$$\beta_{ik}\cdot\alpha_{kj}=\begin{cases}\begin{pmatrix}|a_{jk}b_{ki}| & 0\\ 0 & |a_{jk}b_{ki}|\end{pmatrix} & \text{if } a_{jk}b_{ki}\geq 0,\\[2ex] \begin{pmatrix}0 & |a_{jk}b_{ki}|\\ |a_{jk}b_{ki}| & 0\end{pmatrix} & \text{if } a_{jk}b_{ki}\leq 0.\end{cases}$$

Since the blocks γ_{ij} are either of the form $\left(\begin{smallmatrix}\ell & 0\\ 0 & \ell\end{smallmatrix}\right)$ or of the form $\left(\begin{smallmatrix}0 & \ell\\ \ell & 0\end{smallmatrix}\right)$ for some non-negative integer ℓ, a necessary condition for $\gamma'_{ij}=\gamma_{ij}$ is that all block summands $\beta_{ik}\cdot\alpha_{kj}$ are of the same form, i.e., all of the form $\left(\begin{smallmatrix}\ell & 0\\ 0 & \ell\end{smallmatrix}\right)$ or all of the form $\left(\begin{smallmatrix}0 & \ell\\ \ell & 0\end{smallmatrix}\right)$. This is equivalent to the condition that all the nonzero terms $a_{jk}b_{ki}$ are of the same sign for $k=1,\cdots,n$. On the other hand, once we have that all the nonzero terms $a_{jk}b_{ki}$ are of the same sign, it is easy to see that we will automatically have $\gamma'_{ij}=\gamma_{ij}$. Therefore, we conclude that $(f_A\circ f_B)^*=f_B^*\circ f_A^*$ as maps on $\mathrm{CDiv}_T(X)$ if and only if the condition ($\star$) holds again. We can summarize the above discussion as the following proposition.

Proposition 7.2. *$(f_A \circ f_B)^* = f_B^* \circ f_A^*$ for T-invariant divisors if and only if $(f_A \circ f_B)^* = f_B^* \circ f_A^*$ for divisor classes in the Picard group, if and only if the condition $(\star)$ is satisfied.* $\square$

7.4. **Stability of rational maps on $(\mathbb{P}^1)^n$.** In the last two subsections of this paper, we will leave the realm of monomial maps, and look at some general phenomenon about algebraic stability of rational maps on $(\mathbb{P}^1)^n$. Assume that we have a rational map $f : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ given by

$$f(\mathbf{z}) = f(z_1, \cdots, z_n) = \Big(\frac{p_1(\mathbf{z})}{q_1(\mathbf{z})}, \cdots, \frac{p_n(\mathbf{z})}{q_n(\mathbf{z})}\Big),$$

where the p_j, q_j are polynomials in $\mathbf{z} = (z_1, \cdots, z_n)$ for $i = 1, \cdots, n$. We can assume that p_j and q_j are pairwise relatively prime, otherwise we can divide them by their greatest common divisor. The map f induces a rational map, also denoted by f, on $(\mathbb{P}^1)^n$, in the following way.

$$\begin{aligned} &f([x_1; y_1], \cdots, [x_n; y_n]) \\ = &\Big([P_1(x_i, y_i); Q_1(x_i, y_i)], \cdots, [P_n(x_i, y_i); Q_n(x_i, y_i)]\Big). \end{aligned}$$

Here the P_j and Q_j are obtained by homogenizing the polynomials p_j, q_j in the j-th component of f, with respect to every pair of variables (x_i, y_i), by setting $z_i = x_i/y_i$. We will use $P_1(x_i, y_i)$ as a shorthand for $P_1(x_1, y_1, \cdots, x_n, y_n)$. The concrete formulae for P_j and Q_j are

$$\begin{aligned} P_j(x_i, y_i) &= P_j(x_1, y_1, \cdots, x_n, y_n) \\ &= \Big(\prod_{i=1}^n y_i^{\max\{\deg_{z_i}(p_j), \deg_{z_i}(q_j)\}}\Big) \cdot p_j(\frac{x_1}{y_1}, \cdots, \frac{x_n}{y_n}), \\ Q_j(x_i, y_i) &= Q_j(x_1, y_1, \cdots, x_n, y_n) \\ &= \Big(\prod_{i=1}^n y_i^{\max\{\deg_{z_i}(p_j), \deg_{z_i}(q_j)\}}\Big) \cdot q_j(\frac{x_1}{y_1}, \cdots, \frac{x_n}{y_n}). \end{aligned}$$

Thus the polynomials P_j and Q_j are homogeneous in each pair of variables (x_i, y_i), of the same degree $= \max\{\deg_{z_i}(p_j), \deg_{z_i}(q_j)\}$. We denote this degree by $\deg_{(x_i,y_i)} P_j = \deg_{(x_i,y_i)} Q_j$, and call such polynomials P_j and Q_j *multi-homogeneous.*

Conversely, given $2n$ multi-homogeneous polynomials $P_j(x_i, y_i)$, $Q_j(x_i, y_i)$, $j = 1, \cdots, n$, which are pairwise relatively prime, and satisfy $\deg_{(x_i,y_i)} P_j = \deg_{(x_i,y_i)} Q_j$ for all $i, j = 1, \cdots, n$. Then they induce a rational map $f : (\mathbb{P}^1)^n \dashrightarrow (\mathbb{P}^1)^n$ by sending $([x_1; y_1], \cdots, [x_n; y_n])$ to

$$([P_1(x_i, y_i); Q_1(x_i; y_i)], \cdots, [P_n(x_i, y_i); Q_n(x_i; y_i)]).$$

The indeterminacy set of the rational map f is given by $I_f = \cup_{j=1}^n I_{f,j}$, where $I_{f,j}$ is the set defined by the equations $P_j = Q_j = 0$.

Recall that the Picard group of $(\mathbb{P}^1)^n$ is

$$\mathrm{Pic}((\mathbb{P}^1)^n) = \mathbb{Z} \cdot [D_1] \oplus \cdots \oplus \mathbb{Z} \cdot [D_n],$$

where D_i is the divisor defined by $x_i = 0$, and $[D_i]$ is the linear equivalence class of D_i in $\mathrm{Pic}((P^1)^n)$. Therefore, the pull back map $f^* : \mathrm{Pic}((\mathbb{P}^1)^n) \to \mathrm{Pic}((\mathbb{P}^1)^n)$, with respect to the ordered basis $[D_1], \cdots, [D_n]$, is represented by the matrix

$$\begin{aligned}\mathrm{Deg}(f) &= \Big(\max\{\deg_{z_i}(p_j), \deg_{z_i}(q_j)\}\Big)_{1\le i\le n; 1\le j\le n.}\\ &= \Big(\deg_{(x_i,y_i)}(P_j)\Big)_{1\le i\le n; 1\le j\le n.}\\ &= \Big(\deg_{(x_i,y_i)}(Q_j)\Big)_{1\le i\le n; 1\le j\le n.}\end{aligned}$$

Therefore, the condition $(f^*)^n = (f^n)^*$ for algebraic stability can be translated into the condition $\mathrm{Deg}(f)^n = \mathrm{Deg}(f^n)$. We will give a geometric characterization of algebraic stable maps on $(\mathbb{P}^1)^n$. Before doing that, we need to introduce some notations and facts about $(\mathbb{P}^1)^n$.

Suppose we equip $(\mathbb{C}^2)^n$ with the coordinate $(x_1, y_1, \cdots, x_n, y_n)$, and let $E_j = \{x_j = y_j = 0\} \subset (\mathbb{C}^2)^n$, then there is a quotient map

$$\begin{aligned}\pi : (\mathbb{C}^2)^n \backslash (\cup_{j=1}^n E_j) &\longrightarrow (\mathbb{P}^1)^n\\ (x_1, y_1, \cdots, x_n, y_n) &\longmapsto ([x_1; y_1], \cdots, [x_n; y_n])\end{aligned}$$

For each point $\mathbf{x} \in (\mathbb{P}^1)^n$, the fibre $\pi^{-1}(\mathbf{x})$ is an algebraic torus $(\mathbb{C}^*)^n$.

Suppose that a rational map $f : (\mathbb{P}^1)^n \dashrightarrow (\mathbb{P}^1)^n$ is given by P_j and Q_j, as described above. We can *lift* the rational map f to obtain a polynomial map $F : (\mathbb{C}^2)^n \to (\mathbb{C}^2)^n$. F is defined by the same polynomials P_j and Q_j as f, i.e.,

$$F(x_i, y_i) = \big(P_1(x_i, y_i), Q_1(x_i, y_i), \cdots, P_n(x_i, y_i), Q_n(x_i, y_i)\big).$$

Notice that, a point $\mathbf{x} \in (\mathbb{P}^1)^n$ is in the indeterminacy set I_f if and only if $F(\pi^{-1}(\mathbf{x})) \subset (\cup_{j=1}^n E_j)$. When this happens, since $\pi^{-1}(\mathbf{x})$ is irreducible (in the Zariski topology), we must have $F(\pi^{-1}(\mathbf{x})) \subset E_j$ for some j. To conclude, we have

$$\mathbf{x} \in I_f \iff F(\pi^{-1}(\mathbf{x})) \subset E_j \text{ for some } j.$$

A hypersurface $V \subset (\mathbb{P}^1)^n$ is defined by a multi-homogeneous polynomial $\varphi = \varphi(x_1, y_1, \cdots, x_n, y_n) = 0$. We can consider the *lifting of* V in $(\mathbb{C}^2)^n$, defined by $\widetilde{V} = \overline{\pi^{-1}(V)}$. $\widetilde{V}$ is a hypersurface in $(\mathbb{C}^2)^n$, and the defining equation for $\widetilde{V}$ is also $\varphi = 0$. Notice that V is irreducible in the Zariski topology on $(\mathbb{P}^1)^n$ if and only if $\widetilde{V}$ is irreducible in the Zariski topology on $(\mathbb{C}^2)^n$. This is because if we can factor $\varphi = \varphi_1 \cdot \varphi_2$, then both φ_1 and φ_2 have to be multi-homogeneous.

The following proposition and theorem characterize, geometrically, the algebraic stable maps on $(\mathbb{P}^1)^n$. The proof is a modification of the method used to prove a similar proposition on $\mathbb{P}^n$ by Fornaess and Sibony ([5], see also [11, Proposition 1.4.3]). Also, the results were already given, in the more

general context of multiprojective spaces, by Favre and Guedj [4, Proposition 1.7]. We include them here for completeness.

Proposition 7.3. *For two rational maps $f, g : (\mathbb{P}^1)^n \dashrightarrow (\mathbb{P}^1)^n$, the relation* $\mathrm{Deg}(f \circ g) = \mathrm{Deg}(g) \cdot \mathrm{Deg}(f)$ *holds if and only if there is no hypersurface $V \subset (\mathbb{P}^1)^n$ such that $g(V \backslash I_g) \subset I_f$.*

Proof. First, if there is such a V, we can assume that V is irreducible. Then $U = \pi^{-1}(V \backslash I_g)$ is a nonempty open subset of $\widetilde{V}$, hence is dense in $\widetilde{V}$ and is irreducible. The condition $g(V \backslash I_g) \subset I_f$ means that for all $y \in U$, we have $F(G(y)) \in E_j$ for some j. A priori the E_j may depend on y, but since U is irreducible, this implies that $F(G(U)) \subset E_j$ for some j. Without loss of generality, assume $j = 1$. Furthermore, since U is open and dense in $\widetilde{V}$, and E_1 is a closed subset of $(\mathbb{P}^1)^n$, we conclude that $F(G(\widetilde{V})) \subset E_1$ as well.

Suppose V is defined by the multi-homogeneous polynomial φ, and for $\mathbf{x} \in (\mathbb{P}^1)^n$, the maps f, g are given by

$$f(\mathbf{x}) = \Big([P_1(\mathbf{x}); Q_1(\mathbf{x})], \cdots, [P_n(\mathbf{x}); Q_n(\mathbf{x})]\Big),$$
$$g(\mathbf{x}) = \Big([P'_1(\mathbf{x}); Q'_1(\mathbf{x})], \cdots, [P'_n(\mathbf{x}); Q'_n(\mathbf{x})]\Big).$$

The j-th component of the composition map $f \circ g$ is given by the polynomials $P''_j = P_j(P'_1, Q'_1, \cdots, P'_n, Q'_n)$ and $Q''_j = Q_j(P'_1, Q'_1, \cdots, P'_n, Q'_n)$. A computation on degree shows that

$$\begin{aligned} \deg_{(x_i,y_i)}(P''_j) &= \deg_{(x_i,y_i)}(Q''_j) \\ &= \sum_{k=1}^{n} \deg_{(x_i,y_i)}(P'_k) \cdot \deg_{(x_k,y_k)}(P_j). \end{aligned}$$

This is the (i, j)-th component of the product of matrices $\mathrm{Deg}(g) \cdot \mathrm{Deg}(f)$. On the other hand, $F(G(\widetilde{V})) \subset E_1$ implies that φ divides both polynomials P''_1 and Q''_1. Thus, for some i such that $\deg_{(x_i,y_i)}(\varphi) > 0$, the $(i, 1)$-th component of the matrix $\mathrm{Deg}(f \circ g)$ will be strictly less than the $(i, 1)$-th component of the product of matrices $\mathrm{Deg}(g) \cdot \mathrm{Deg}(f)$. The two matrices cannot be equal.

Conversely, it is easy to see that if there is no such hypersurface, then the polynomials P''_j and Q''_j will be pairwise relatively prime, with the desired degrees. Hence we will have $\mathrm{Deg}(f \circ g) = \mathrm{Deg}(g) \cdot \mathrm{Deg}(f)$. □

Theorem 7.4. *A rational map $f : (\mathbb{P}^1)^n \dashrightarrow (\mathbb{P}^1)^n$ is algebraically stable if and only if there does not exist an integer k and a hypersurface $V \subset (\mathbb{P}^1)^n$ such that $f^k(V \backslash I_{f^k}) \subset I_f$.*

Proof. This is a direct consequence of the Proposition 7.3 by using an induction argument on k and setting $g = f^k$ in the proposition. □

7.5. **Stability of polynomial maps on** $(\mathbb{P}^1)^n$**.** Recall that for $f(\mathbf{z}) = (p_1(\mathbf{z})/q_1(\mathbf{z}), \cdots, p_n(\mathbf{z})/q_n(\mathbf{z}))$, it induces a rational map on $(\mathbb{P}^1)^n$, and the pull back f^* is represented by the matrix

$$\mathrm{Deg}(f) = \Big(\max\{\deg_{z_i}(p_j), \deg_{z_i}(q_j)\}\Big)_{1\le i\le n; 1\le j\le n.}$$

In particular, if $f = (f_1, \cdots, f_n)$ is a polynomial map, then $p_j = f_j$ and $\deg_{z_i}(q_j) = 0$ for all i, j, hence

$$\mathrm{Deg}(f) = \Big(\deg^+_{z_i}(f_j)\Big).$$

Here $\deg^+_{z_i}(f_j) = \max\{\deg_{z_i}(f_j), 0\}$ is almost the degree of f; the only difference is that for the zero polynomial, we have $\deg^+_{z_i}(0) = 0$ instead of the usual convention $\deg_{z_i}(0) = -\infty$.

Given a polynomial map f, for notational simplicity, we will denote $\deg^+_{z_i}(f_j)$ by $\deg_i(f_j)$ and just call it *the degree of* f_j with respect to z_i, and we will call $\mathrm{Deg}(f)$ the *degree matrix* of f.

Our next goal is to proof the following Theorem 7.5, which gives a family of algebraically stable polynomial maps on $(\mathbb{P}^1)^n$, and a partial converse which gives a characterization for algebraically stable polynomial maps on $(\mathbb{P}^1)^n$ under certain condition. First, we need to define some terminologies.

Definition. For a polynomial $h \in \mathbb{C}[z_1, \cdots, z_n]$ and a monomial $\mu = z_1^{a_1} \cdots z_n^{a_n}$, we said that μ is a *monomial term* of h if the coefficient of μ in h is not zero. A monomial term μ of h is said to be *the dominant term* of h if for all $i = 1, \cdots, n$, we have $\deg_i(h) = \deg_i(\mu)$.

Equivalently, a monomial term $\mu = z_1^{a_1} \cdots z_n^{a_n}$ of h is the dominant term of h if and only if, for all monomial term $z_a^{b_1} \cdots z_n^{b_n}$ of h , we have $a_i \ge b_i$ for $i = 1, \cdots, n$. For example, the polynomial $h = 2z_1^2 z_2 + 3z_1^2 + z_1 z_2 - 5z_2 - 1$ has a dominant term $z_1^2 z_2$. Notice that not all polynomials have a dominant term. For example, $h = z_1 + z_2$ does not have a dominant term.

Theorem 7.5. *Let $f = (f_1, \cdots, f_n)$ be a polynomial map.*

(1) *If each f_j is dominated by a monomial term, then f is algebraically stable on $(\mathbb{P}^1)^n$.*
(2) *Assume that, for some iterate $f^N = (f_1^{(N)}, \cdots, f_n^{(N)})$ of f, we have $\deg_{z_i}(f_j^{(N)}) > 0$ for all $i, j = 1, \cdots, n$. Then f being algebraically stable on $(\mathbb{P}^1)^n$ implies that each f_j must have a dominant term.*

We will prove the theorem in steps. First, observe that, if each f_j has a dominant term $\mu_j = z_1^{a_{1j}} \cdots z_n^{a_{nj}}$, then we know $\deg_i(f_j) = a_{ij}$, and therefore $\mathrm{Deg}(f) = (a_{ij})$.

Next, if $f = (f_1, \cdots, f_n)$ and $g = (g_1, \cdots, g_n)$ are two polynomial maps, such that each of the f_j and g_k has a dominant term, say $\mu_j = z_1^{a_{1j}} \cdots z_n^{a_{nj}}$ and $\nu_k = z_1^{b_{1k}} \cdots z_n^{b_{nk}}$, respectively. Consider the j-th component of the

composition $fg = f \circ g$. It will be of the following form:

$$\begin{aligned}(fg)_j &= f_j(g_1, \cdots, g_n) \\ &= c_j \cdot \mu_j(\nu_1, \cdots, \nu_n) + \{\text{lower order terms}\} \\ &= c_j \cdot \nu_1^{a_{1j}} \cdots \nu_n^{a_{nj}} + \{\text{lower order terms}\} \\ &= c_j \cdot \prod_{k=1}^{n} \left(z_1^{b_{1k}} \cdots z_n^{b_{nk}} \right)^{a_{kj}} + \{\text{lower order terms}\}\end{aligned}$$

where c_j is some constant. That is, $\mu_j(\nu_1, \cdots, \nu_j)$ is the dominant term of $(fg)_j$, and the degree

$$\deg_i((fg)_j) = \sum_{k=1}^{n} b_{ik} a_{kj},$$

which is the (i,j)-th component of the product of matrices $\mathrm{Deg}(g) \cdot \mathrm{Deg}(f)$. We summarize the above discussion as the following proposition.

Proposition 7.6. *If $f = (f_1, \cdots, f_n)$ and $g = (g_1, \cdots, g_n)$ are two polynomial maps, such that each of the f_j and g_j has a dominant term, then the composition (fg) is also a polynomial map such that each component has a dominant term. Furthermore, for the degree matrix of (fg), we have*

$$\mathrm{Deg}(fg) = \mathrm{Deg}(g) \cdot \mathrm{Deg}(f).$$

□

As a corollary of the proposition, we can now prove the first part of Theorem 7.5.

Corollary. *If $f = (f_1, \cdots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map, and if each f_j has a dominant term, then f is algebraically stable on $(\mathbb{P}^1)^n$.*

Proof. We know that for all k, the iterate f^k is also a polynomial map such that every component has a dominant term. Hence an induction argument shows that

$$\mathrm{Deg}(f^k) = (\mathrm{Deg}(f))^k.$$

Since the degree matrix represents f^* with respect to the ordered basis $[D_1], \cdots, [D_n]$, we also have

$$(f^k)^* = (f^*)^k.$$

□

Corollary. *If $f = (f_1, \cdots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map, and if each f_j has a dominant term, then the first dynamical degree of f is an algebraic integer.*

Proof. We know that f is algebraically stable, and f^* is represented by the degree matrix $\mathrm{Deg}(f)$. The first dynamical degree of f is the spectral radius of the degree matrix, an integer matrix. Hence the first dynamical degree is an algebraic integer. □

There is a conjecture proposed by Bellon and Viallet (see [8, Conjecture 1.1]), namely, the first dynamical degree of every rational map is an algebraic integer. The corollary shows that the conjecture holds for the case of polynomial maps with a dominant term for each component.

Notice that every monomial polynomial has a dominant term, namely the monomial itself. Hence we obtain another proof that every monomial polynomial map is algebraically stable on $(\mathbb{P}^1)^n$.

Now we turn to the second part of Theorem 7.5.

Proposition 7.7. *Let $f_1, \cdots, f_n, g_1, \cdots g_n \in \mathbb{C}[z_1, \cdots, z_n]$ be polynomials such that $\deg_i(g_j) > 0$ for all $i, j = 1, \cdots, n$. They induce polynomial maps $f = (f_1, \cdots, f_n)$ and $g = (g_1, \cdots, g_n) : \mathbb{C}^n \to \mathbb{C}^n$. If we have*

$$\mathrm{Deg}(fg) = \mathrm{Deg}(g) \cdot \mathrm{Deg}(f),$$

then every component f_j must have a dominant monomial term.

Proof. Assume otherwise, i.e., some f_j does not have a dominant term, we want to show the two matrices are different. Under the assumption, for every monomial term $\mu = z_1^{a_1} \cdots x_n^{a_n}$, there is some $a_k < \deg_k(f_j)$; without loss of generality we can assume $a_1 < \deg_1(f_j)$. Consider the composition $\mu \circ g = \mu(g_1, \cdots, g_n)$, it is easy to see that, for all $i = 1, \cdots, n$,

$$\begin{aligned}\deg_i(\mu \circ g) &= a_1 \cdot \deg_i(g_1) + \cdots + a_n \cdot \deg_i(g_n) \\ &< \deg_1(f_j) \cdot \deg_i(g_1) + \cdots + \deg_n(f_j) \cdot \deg_i(g_n)\end{aligned}$$

The inequality is strict because $a_1 < \deg_1(f_j)$ and $\deg_i(g_1) > 0$. Therefore, we know

$$\deg_i(f_j \circ g) < \deg_1(f_j) \cdot \deg_i(g_1) + \cdots + \deg_n(f_j) \cdot \deg_i(g_n),$$

Notice that the term on the left is the (i, j)-th entry of the matrix $\mathrm{Deg}(fg)$, whereas the term on the right is the (i, j)-th entry of the matrix $\mathrm{Deg}(g) \cdot \mathrm{Deg}(f)$. Hence, the two matrices are different. $\square$

The condition $\deg_i(g_j) > 0$ in the proposition is essential. For example, let $f(z_1, z_2, z_3) = g(z_1, z_2, z_3) = (z_2, z_3, z_1 + z_2)$, then we still have $\mathrm{Deg}(fg) = \mathrm{Deg}(g) \cdot \mathrm{Deg}(f)$, but $f_3 = z_1 + z_2$ does not have a dominant monomial term.

Corollary. *Suppose that $f = (f_1, \cdots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map, and for some iterate $f^N = (f_1^{(N)}, \cdots, f_n^{(N)})$ of f, we have $\deg_i(f_j^{(N)}) > 0$ for all $i, j = 1, \cdots, n$. If $\mathrm{Deg}(f^k) = \mathrm{Deg}(f)^k$ for all k, then every component f_j of f must have a dominant monomial term.*

Proof. Apply f and $g = f^N$ to the proposition. $\square$

This also concludes the proof of Theorem 7.5. $\square$

References

[1] Eric Bedford and Kyounghee Kim, *Linear recurrences in the degree sequences of monomial mappings*, Ergodic Theory Dynam. Systems **28** (2008), no. 5, 1369–1375, available at `arXiv:0710.1642[math.DS]`.

[2] V. I. Danilov, *The geometry of toric varieties*, Russ. Math. Surv. **33** (1978), 97–154.

[3] Charles Favre, *Les applications monomiales en deux dimensions*, Michigan Math. J. **51** (2003), no. 3, 467–475, available at `arXiv:0210025[math.CV]`.

[4] Charles Favre and Vincent Guedj, *Dynamique des applications rationnelles des espaces multiprojectifs*, Indiana Univ. Math. J. **50** (2001), no. 2, 881–934.

[5] John Erik Fornaess and Nessim Sibony, *Complex dynamics in higher dimension. II*, Modern methods in complex analysis, Annals of Math Studies, vol. 137, Princeton University Press, Princeton, NJ, 1995, pp. 135–182.

[6] William Fulton, *Introduction to Toric Varieties*, Annal of Math Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993.

[7] Vincent Guedj, *Propriétés ergodiques des applications rationnelles*, available at `arXiv:0611302[math.CV]`.

[8] Boris Hasselblatt and James Propp, *Degree-growth of monomial maps*, Ergodic Theory Dynam. Systems **27** (2007), no. 5, 1375–1397, available at `arXiv:0604521[math.DS]`.

[9] Mattias Jonsson and Elizabeth Wulcan, *Stabilization of monomial maps*, available at `arXiv:1001.3938[math.DS]`.

[10] Miles Reid, *Graded rings and varieties in weighted projective space*, available at `http://www.warwick.ac.uk/~masda/surf/more/grad.pdf`.

[11] Nessim Sibony, *Dynamics of rational maps on* $\mathbb{P}^n$, Complex dynamics and geometry, SMF/AMS Texts and Monographs, vol. 10, American Mathematical Society, Providence, RI, 2003, pp. 85–166.

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